

A stabilized logical quantum bit encoded in grid states of a superconducting cavity

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The majority of quantum information tasks require error-corrected logical qubits whose coherence times are vastly longer than that of currently available physical qubits. Among the many quantum error correction codes, bosonic codes are particularly attractive as they make use of a single quantum harmonic oscillator to encode a correctable qubit in a hardware-efficient manner. One such encoding, based on grid states of an oscillator, has the potential to protect a logical qubit against all major physical noise processes. By stroboscopically modulating the interaction of a superconducting microwave cavity with an ancillary transmon, we have successfully prepared and permanently stabilized these grid states. The lifetimes of the three Bloch vector components of the encoded qubit are enhanced by the application of this protocol, and

agree with a theoretical estimate based on the measured imperfections of the experiment.

Introduction

In quantum physics, a multi-partite system can be prepared in a zero-entropy *pure state* even though the outcome of any local measurement performed on one of the system components is perfectly random (1). Such a composite system can implement a *fully non-local*, protected logical qubit. To understand how this protection arises, let us first consider the case of a *partially non-local* qubit formed by a particle placed in one of two distant positions, T and B (for Top and Bottom). In the thought experiment represented in Fig. 1a, the state $e^{-i\phi/2} \cos(\theta/2)|T\rangle + e^{i\phi/2} \sin(\theta/2)|B\rangle$ is generated by sending a magnetic spin-1/2 neutral atom polarized along (θ, ϕ) through a magnetic field gradient which deflects it according to its vertical spin component. After this Stern-Gerlach-type device, the phase ϕ gets rapidly randomized due to undesired interactions with spurious degrees of freedom in the environment. These degrees of freedom couple to the logical qubit through the $\mathbf{Z} = |T\rangle\langle T| - |B\rangle\langle B|$ Pauli operator. Such an operator, which reveals the presence of the particle in T or B , is *local*. Interactions with the environment through this operator trigger random \mathbf{Z} gates on the logical qubit known as *phase-flips* ($\phi \rightarrow \phi + \pi$) (2). On the other hand, if the positions T and B are distant enough, direct tunneling between them is suppressed, so any physically realistic Hamiltonian has vanishing values for the matrix elements of the form $|T\rangle\langle B|$ and $|B\rangle\langle T|$. As a consequence, the $\mathbf{X} = |T\rangle\langle B| + |B\rangle\langle T|$ and $\mathbf{Y} = -i|T\rangle\langle B| + i|B\rangle\langle T|$ logical Pauli operators cannot be probed directly and *bit-flips* ($\theta \rightarrow \pi - \theta$) of the logical qubit can be suppressed by trapping the particle in one of two boxes in T and B. In this example, the logical qubit is only partially delocalized and thus only partially protected: the \mathbf{X} and \mathbf{Y} operators are *non-local* in the sense that \mathbf{X} and \mathbf{Y} gates connect distant positions—and thus do not take place directly, offering an opportunity for correction of *bit-flips*—but the \mathbf{Z} operator is *local*—so that *phase-flips* are not correctable. Extending this idea to a fully delocalized qubit, for which all three Pauli operators are non-local,

is at the heart of quantum error correction schemes such as the *Steane code* (3), the *surface code* (4), and the proposed implementations for topological quantum computation (5).

In the same vein as the partially delocalized particle example described above, one can encode a logical qubit over two distant coherent states in position-momentum (q, p) phase-space of a harmonic oscillator. This bosonic code, known as the *Schrödinger cat code* (6–8), provides a “half-protected” qubit. Indeed, if the two coherent states are distant enough in the oscillator phase space, two logical Pauli operators—encoding the phase of the superposition—are non-local, but the third one—encoding which-location information—is local and can be directly probed by a q or p detection. Can we solve this conundrum and design a bosonic code so that all three logical Pauli operators are non-local?

In 2001, Gottesman, Kitaev, and Preskill (GKP) proposed one such entirely non-local bosonic encoding (9). The code states are *grid states*, periodic in phase-space along both q and p (10). Defining displacements as $\mathbf{D}(\beta) = e^{-i\text{Re}(\beta)\mathbf{p} + i\text{Im}(\beta)\mathbf{q}}$, with the normalization $[\mathbf{q}, \mathbf{p}] = i$ and $\langle \mathbf{q}^2 \rangle = \langle \mathbf{p}^2 \rangle = 1/2$ in the ground state of the oscillator, the operators $\mathbf{S}_a = \mathbf{D}(a = 2\sqrt{\pi})$ and $\mathbf{S}_b = \mathbf{D}(b = ia)$ commute. They are the stabilizers of a two-dimensional code defined as their mutual +1 eigenspace. The Pauli operators of the logical qubit are also displacement operators given by $\mathbf{X} = \mathbf{D}(a/2)$, $\mathbf{Z} = \mathbf{D}(b/2)$, and $\mathbf{Y} = \mathbf{D}((a+b)/2)$. They commute with the stabilizers and respect the Pauli group composition rules within the code manifold ($\mathbf{X}^2 = \mathbf{Y}^2 = \mathbf{Z}^2 = \mathbf{I}$ and $\mathbf{XYZ} = i\mathbf{I}$). All of these operators are non-local since they directly connect regions of phase-space separated by at least $a/2$. In contrast with other bosonic codes that have been proposed (11, 12) and experimentally realized (13, 14), the GKP code offers a possible protection against most physical noise processes (9, 15–17).

To understand this protection, we must consider physically realistic logical states, such as those recently experimentally synthesized in the motional degree of freedom of a trapped

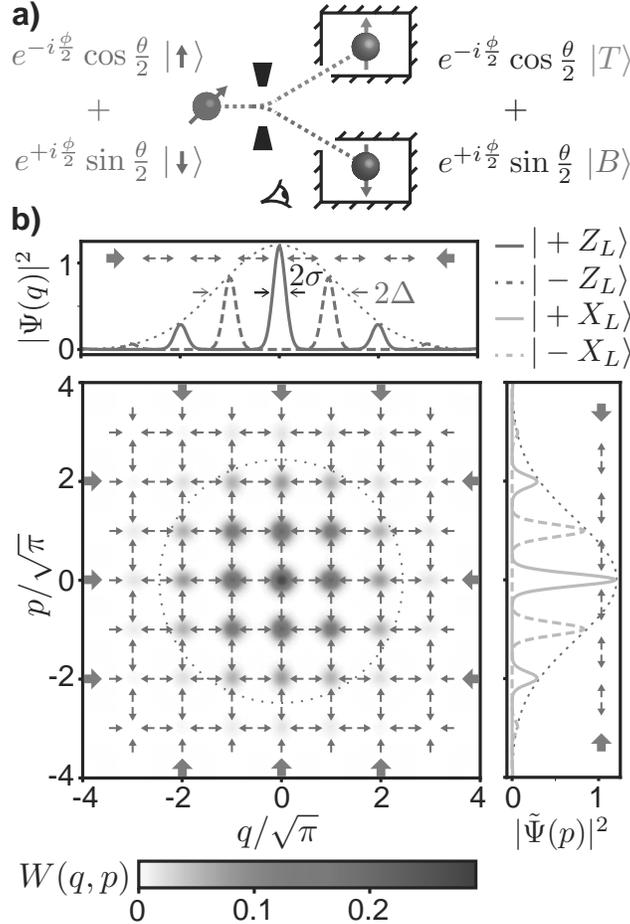


Figure 1: **Non-locally encoded quantum information** **a)** Partially non-local encoding: in a thought experiment, a Stern-Gerlach setup maps a local qubit (spin-1/2 particle) onto a partially non-local qubit (particle confined in one of two boxes). Bit-flips are then suppressed, but a local observer can reveal the particle position, dephasing a superposition of $|T\rangle$ and $|B\rangle$. **b)** Fully non-local encoding: realistic GKP code states are finitely squeezed grid states. The simulated Wigner function of the fully mixed logical state in a code defined by a width $\sigma = 0.25$ for the peaks and $\Delta = 1/(2\sigma) = 2$ for the normalizing envelope is given here as an illustration. Our stabilization protocol entails position and momentum-dependent shifts of the state which prevent the squeezed peaks from spreading (blue arrows) and the overall envelope from extending (purple arrows). Side panels represent the probability distributions of the $|\pm X_L\rangle$ and $|\pm Z_L\rangle$ states along each quadrature, which retain disjoint supports along q or p under stabilization. The GKP qubit is thus fully non-local and protected both against phase-flips and bit-flips.

ion (18). In contrast with the ideal code states which extend infinitely in phase-space, a realistic state $|\Psi_L\rangle$ is only an *approximate* eigenstate of the stabilizers verifying $\langle\Psi_L|\mathbf{S}_{a,b}|\Psi_L\rangle \approx 1$. For the *square code*, such a state is a superposition of squeezed states periodically displaced by a with a Gaussian weight profile (see Fig. 1b). For a balanced code with equal squeezing in both quadratures, the standard deviation σ of each peak in phase-space is related to the width Δ of the normalizing envelope (dashed gray line in Fig. 1b) by $\sigma\Delta = 1/2$. If $\sigma \ll a$, two states within this manifold that are shifted from one another by a along q or p are quasi-orthogonal and form a valid logical qubit. Remarkably, this qubit can be protected against any noise process acting on the system via local operators such as \mathbf{q} , \mathbf{p} , and any finite-order polynomial of these two. These include decoherence processes such as photon relaxation (16, 17) or pure dephasing of the oscillator, and Hamiltonian evolutions such as those induced by spurious nonlinearities. Indeed, when acting on logical GKP states with finite extent, these processes result in a continuous evolution of quasi-probability distribution in phase-space (9, 15, 19). Since each pair of quasi-orthogonal logical qubit states $|\pm X_L\rangle$, $|\pm Y_L\rangle$ and $|\pm Z_L\rangle$ have non-overlapping support in phase-space as shown in Fig. 1b, no flip between these states can occur at short times.

In this framework, error-correction consists of preventing a logical state from shifting far enough to be erroneously decoded as the orthogonal one. Small shifts can be detected without perturbing the logical qubit by measuring the real and imaginary part of the stabilizers, which are commuting Hermitian operators. Indeed, the measurement outcomes only reveal the position of the state in phase-space *modulo* $2\pi/a = \sqrt{\pi}$, which does not depend on the state of the logical qubit. One can then apply a corrective shift to recenter the grid state in $q, p = 0 \bmod 2\pi/a$. The error-correction protocol described in the two next sections is based on such stabilizer measurements and feedback.

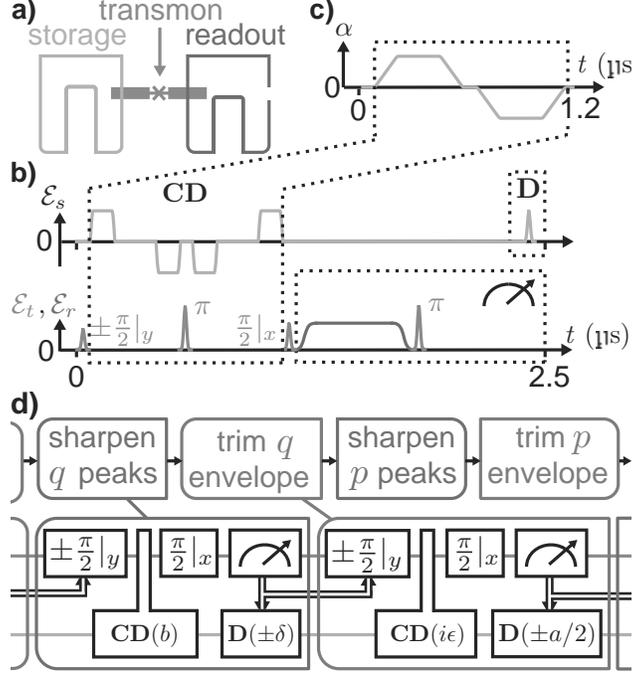


Figure 2: **Code stabilization sequence.** **a)** Schematic of the two coaxial resonators bridged by a single transmon as used in the experiment. **b)** Detailed pulse sequence for a single stabilization round lasting $2.5 \mu\text{s}$. The transmon control pulses (green) include $\pi/2$ -rotations for initialization and readout along the σ_x and σ_y axes (dispersive readout pulse in gray), and stroboscopic π -rotations to cancel the effect of undesired terms in (2). The storage oscillator drive pulses (light pink) temporarily shift the oscillator state by a large value α to implement the conditional displacement CD, and perform the feedback displacement D concluding the round. **c)** The sign of α is inverted at the same time as a π -rotation of the transmon, leading to CD from the second term in (2). **d)** The full stabilization sequence alternates indefinitely two peak-sharpening rounds and two envelope-trimming rounds to prevent spreading of the grid state peaks and envelope in phase-space (respectively blue and purple arrows in Fig. 1). Each round, a conditional displacement entangles the transmon and the storage oscillator. A subsequent measurement of the transmon controls the sign of a feedback shift of the oscillator and of a $\pi/2$ -rotation resetting the transmon (double-stroke black arrows). The peak-sharpening shift $\delta \simeq 0.2$ maximizes the stabilizer value in steady-state. The envelope-trimming conditional displacement by $\epsilon \simeq 0.2$ sets the width of the grid state envelope, which is optimal given experimental constraints (10).

1 Measurement of displacement operators

In our experiment, two reentrant coaxial microwave cavities made out of bulk aluminum are bridged by a single superconducting transmon (20) and are anchored at the base-stage of a dilution refrigerator (see Fig. 2a). The fundamental mode of the first cavity (storage) hosts the GKP grid states and is engineered to have rather long single-photon lifetime $T_s = 245 \mu\text{s}$, while the fundamental mode of the second cavity (readout), overcoupled to a transmission line, performs a quantum non-demolition readout of the transmon in 700 ns (10). The transmon energy and coherence lifetimes are $T_1 = 50 \mu\text{s}$ and $T_2 = 60 \mu\text{s}$. The storage mode and the transmon are dispersively coupled, and in the doubly rotating frame at these two modes resonance frequencies, their joint Hamiltonian is

$$\frac{\mathbf{H}(t)}{\hbar} = -\frac{\chi}{2} \mathbf{a}^\dagger \mathbf{a} \sigma_z + i\mathcal{E}_s(t) \mathbf{a}^\dagger - i\mathcal{E}_s^*(t) \mathbf{a}, \quad (1)$$

where $\mathbf{a} = (\mathbf{q} + i\mathbf{p})/\sqrt{2}$ is the annihilation operator of the storage mode, σ_z is a Pauli operator of the transmon, $\chi/2\pi = 28 \text{ kHz}$ is the dispersive shift, and the displacement rate \mathcal{E}_s results from a resonant microwave drive applied to the storage mode through a weakly coupled port. One can mathematically cancel this drive term by moving to a displaced frame via the transformation $\mathbf{a} \rightarrow \mathbf{a} + \alpha(t)$, where $\alpha(t) = \int_0^t \mathcal{E}_s(t') dt'$ is the response of the storage oscillator in absence of the transmon (8). In this frame, the Hamiltonian now reads

$$\frac{\tilde{\mathbf{H}}(t)}{\hbar} = -\frac{\chi}{2} \mathbf{a}^\dagger \mathbf{a} \sigma_z - \frac{\chi}{2} (\alpha(t) \mathbf{a}^\dagger + \alpha^*(t) \mathbf{a}) \sigma_z - \frac{\chi}{2} |\alpha(t)|^2 \sigma_z. \quad (2)$$

In this expression, the first term leads to a rotation of the storage mode conditioned on the transmon state, the second term creates a displacement of the storage mode conditioned on the transmon state, and the third term gives a rotation of the transmon around the σ_z axis. In order to measure a displacement operator $\mathbf{D}(\beta)$ of the storage mode, we perform the sequence depicted in Fig. 2b-c. This sequence, akin to the measurement scheme introduced in (21), cancels the effect of the first and last terms while retaining the effect of the middle one. After preparing the transmon in $|+x\rangle$, the storage mode is displaced by a large value α (up to

$|\alpha|^2 \simeq 1000$ photons in the laboratory frame in our experiment) and brought back to the origin of phase-space after an arbitrary time. Next, the transmon state is flipped and the oscillator state is displaced in the opposite direction for the same duration. Overall, this results in the oscillator state being displaced conditioned on the transmon state following the unitary evolution $\mathbf{CD}(\beta) = e^{i(-\text{Re}(\beta)\mathbf{p} + \text{Im}(\beta)\mathbf{q})\frac{\sigma_z}{2}}$ for $\beta = i\sqrt{2}\chi \int_t \alpha_+(t)dt$, where $\alpha_+ = \alpha$ for the first half of the evolution, and $\alpha_+ = -\alpha$ for the second half, which accounts for the changing sign of σ_z when the transmon is flipped. In this evolution, we have neglected a small deterministic displacement of the storage mode resulting from the non-commutation of $\tilde{\mathbf{H}}$ at different times (10).

This conditional displacement can equivalently be viewed as a rotation of the transmon Bloch vector around the σ_z axis by an angle dependent on the phase-space distribution of the storage mode. Measuring the transmon along σ_x or σ_y then yields up to one bit of information about this distribution. More precisely, after the conditional displacement $\mathbf{CD}(\beta)$, we show that $\langle \sigma_x - i\sigma_y \rangle = \langle \mathbf{D}(\beta) \rangle$ (10, 22). As a consequence, if the storage mode is in the manifold of $\mathbf{D}(\beta)$ with eigenvalue $e^{i\lambda}$, the transmon is in the pure state whose Bloch vector forms an angle λ with σ_x on the equator. Measuring the transmon along a varying angle, resetting it (23, 24), and repeating the sequence until λ is found thus performs *phase-estimation* of $\mathbf{D}(\beta)$ (25, 26). Several schemes applying this procedure to $\beta = a, b$ were proposed to sequentially prepare ideal grid states (15, 27–31). In the next section, we describe how to circumvent the complex calculations or heavy post-selection required by these schemes by using a proportional-like, Markovian feedback to permanently stabilize the finitely squeezed GKP code.

Conveniently, conditional displacements can also be employed to obtain the expectation value $\langle \mathbf{D}(\beta) \rangle$ of any displacement operator for an arbitrary state of the storage oscillator. This leads precisely to the state characteristic function (32), which is the two-dimensional Fourier transform of the Wigner function, $C(\beta) = \langle \mathbf{D}(\beta) \rangle = \iint W(q, p) e^{-i\text{Re}(\beta)p + i\text{Im}(\beta)q} dq dp$. This complex-valued representation fully characterizes an arbitrary state. In our experiment, we mea-

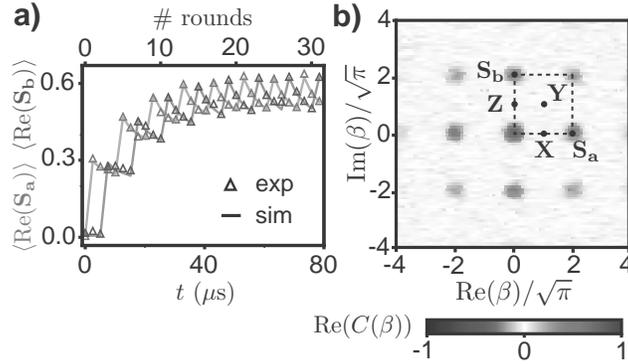


Figure 3: **Stabilized square code in steady-state.** **a)** Measured average value of the code stabilizers real part when turning on stabilization from the vacuum state. Each stabilizer oscillates over a 4-round period as a result of the periodicity of the stabilization sequence (see Fig. 2d), and steady-state is reached in about 20 rounds. The maximum value of $\text{Re}(\mathbf{S}_{a,b})$ for this finitely squeezed code is 0.86 (10). **b)** Real part of the measured characteristic function of the storage mode in steady-state (after 200 rounds stabilization). Particular points corresponding to stabilizers and Pauli operators are indicated by black circles, and the dashed lines enclose an area of 4π .

sure $\text{Re}(C(\beta))$, which contains the information about the symmetric component of the Wigner function, to characterize the generated grid states as represented on Fig. 3–6. The imaginary part $\text{Im}(C(\beta))$ contains information about the anti-symmetric component of the Wigner function and is expected to take a uniform null value for the symmetric grid states we consider. We have verified this property at critical points (10).

2 Stabilization of the GKP code manifold

Given experimental constraints, we aim to stabilize a finite size GKP code with envelope width $\Delta \simeq 3.2$, chosen to maximize the logical qubit coherence time (10). Given this envelope, a state within the manifold has a peak width satisfying $\sigma = 1/(2\Delta) = 0.16$, and we verify that a pair of encoded Clifford states at opposite points on the logical Bloch sphere have an overlap bounded by $|\langle +Y_L | -Y_L \rangle|^2 = 4 \times 10^{-7}$ (10). Thus, the logical qubit is rigorously defined in this finitely squeezed code, for which \mathbf{S}_a , \mathbf{S}_b , \mathbf{X} , \mathbf{Y} and \mathbf{Z} are only *approximate* stabilizers and Pauli

operators. Photon dissipation and other decoherence mechanisms tend to diffusively shift the oscillator state, which, on average, results in a state with off-centered and broader peaks (width $\tilde{\sigma} > \sigma$) (10). Efficient quantum error-correction then requires two conditions. First, we should keep the oscillator state probability distribution peaked in phase-space at q , $p = 0 \bmod 2\pi/a$ and require $\tilde{\sigma} \ll a/2$ at all times to avoid flips of the logical qubit. Second, we should prevent the grid-state envelope from drifting or expanding beyond Δ in phase-space.

To satisfy the first condition, we repeatedly corral the oscillator state into an approximate $+1$ eigenstate of the stabilizers by repeating elementary *peak-sharpening rounds*. During one of these rounds, we acquire one bit of information on the stabilizer \mathbf{S}_a (or \mathbf{S}_b), which encodes the state coordinate $p \bmod 2\pi/a$ (respectively $q \bmod 2\pi/a$) in phase space. Following this measurement, we apply feedback to the oscillator by shifting its state in p (respectively q) based on this value only (see Fig. 2d). More precisely, for a p -peak sharpening round, the transmon is prepared in $|+x\rangle$ and the storage mode state conditionally displaced by $\mathbf{CD}(a)$ so that $\langle \sigma_x - i\sigma_y \rangle = \langle \mathbf{S}_a \rangle$. In the limit that the state is close enough to the coding manifold prior to the stabilization round ($\langle \mathbf{S}_a \rangle \simeq +1$), measuring σ_x does not yield any information at first order. On the other hand, measuring σ_y provides one bit of information as to whether the grid state is more likely to be positioned to the left ($\text{Im}(\mathbf{S}_a) < 0$) or right ($\text{Im}(\mathbf{S}_a) > 0$) of $p \bmod 2\pi/a = 0$. The backaction of such a single-round measurement of $\text{Im}(\mathbf{S}_a)$ partly collapses the peaks of the p -distribution in the corresponding direction (see (10) for the explicit form of the associated Kraus operators and details on the oscillator state evolution). This sharpens the peaks as desired, and shifts each peak center of mass, which we correct by applying a feedback displacement of $\pm\delta \simeq \pm 0.2$ to the oscillator, where the sign is determined by the σ_y measurement result. The displacement length δ is chosen to compensate for the backaction-induced shift in the steady-state of stabilization, and thus maximize the stabilizer expectation value (10). Continuously repeating such p peak-sharpening rounds, and similarly arranged q peak-sharpening rounds, results in position and momentum dependent shifts of the quasi-probability distribution

as schematized by blue arrows in Fig. 1b. These shifts anchor the grid state to the center of phase-space and prevent the peaks from spreading.

In order to meet the second condition and maintain the desired code extension in phase-space, we regularly trim the grid state envelope by a controlled amount with a modified version of this feedback protocol (see Fig. 2d). This is needed as the conditional displacements applied during peak-sharpening expand the envelope, which increases sensitivity to dissipation (10). In the stabilization sequence, we thus interleave peak-sharpening rounds with *envelope-trimming rounds*, for which measurement of the imaginary part of the stabilizers is replaced by that of much shorter displacements $\mathbf{D}(\epsilon)$ and $\mathbf{D}(i\epsilon)$, with $\epsilon \simeq 0.2$. Just as the peak-sharpening rounds cause position and momentum dependent shifts of the oscillator state with a period $2\pi/a$ in phase-space (blue arrows in Fig. 1b), the shifts induced by envelope-trimming rounds follow a much larger periodicity of $2\pi/\epsilon$ (purple arrows). They effectively confine the probability distribution to a region of phase-space centered at the origin. The displacement length ϵ sets the envelope width in steady-state (10). The feedback shifts in these envelope-trimming rounds are chosen to be exactly $\pm a/2$, which translates the grid state center of mass without modifying the stabilizer values.

Starting from the ground state of the oscillator, we apply indefinitely the full stabilization protocol alternating measurement of 4 different displacement operators as summarized in Fig. 2d. In Fig. 3a we plot the measured average values of $\text{Re}(\mathbf{S}_a)$ and $\text{Re}(\mathbf{S}_b)$ after n rounds of stabilization. The stabilizer values increase rapidly to converge to a steady-state in about 20 rounds. On top of this trend, each stabilizer mean value oscillates over a period of 4 rounds by increasing to 0.62 when the peaks are sharpened in the corresponding phase-space quadrature and then decaying down to 0.5 over the 3 next rounds. Beyond this periodic oscillation, the stabilizers do not evolve over hundreds of rounds (not shown), which indicates that our protocol fully stabilizes the code manifold.

We plot the real part of the characteristic function of the steady state after 200 rounds of stabilization in Fig. 3b. This state is a maximally mixed state of the logical qubit, as can be seen from the null value of the points corresponding to all three logical Pauli operators. Note that this characteristic function representation is the Fourier conjugate to the theoretical Wigner representation given in Fig. 1, even though the two are similar for grid states. These results are quantitatively reproduced by master-equation simulations (10) (lines in Fig. 3a) whose parameters are all independently calibrated. From these simulations, we estimate that the squeezing of the peaks of the generated grid states oscillates between 7.4 dB and 9.5 dB in steady-state—close to the level required for fault-tolerant quantum computation with a next-level of quantum error-correction (33–35)—and the average photon number oscillates between 8.6 and 10.2 photons.

3 Logical qubit initialization

Once steady-state of the stabilization has been reached, the oscillator is approximately in the code manifold and we may initialize or readout the logical qubit by replacing one of the stabilization rounds with a measurement of the real part of one of the ideal, infinitely squeezed code Pauli operators \mathbf{X} , \mathbf{Y} and \mathbf{Z} . These operators, whose complex eigenvalues lie on the unit circle, have a sharply peaked distribution around +1 and -1 so that this initialization can be done with a single-round measurement. To measure $\text{Re}(\mathbf{X})$, $\text{Re}(\mathbf{Y})$, or $\text{Re}(\mathbf{Z})$, we initialize the transmon in $|+x\rangle$ and apply the conditional displacement $\mathbf{CD}(\beta)$ with $\beta = a/2, (a+b)/2, b/2$ respectively. This gate is followed by the displacement $\mathbf{D}(-\beta/2)$ in order for the whole sequence to shift the grid state by 0 or β rather than $\pm\beta/2$, which would offset the grid state by half a lattice period. After the sequence, $\langle\sigma_x - i\sigma_y\rangle = \langle\mathbf{X}\rangle, \langle\mathbf{Y}\rangle$ or $\langle\mathbf{Z}\rangle$ and a subsequent σ_x readout of the transmon with outcome ± 1 heralds the preparation of the approximate $|\pm X_L\rangle, |\pm Y_L\rangle$ or $|\pm Z_L\rangle$ state.

However, the logical qubit cannot be perfectly initialized with a single-round measurement

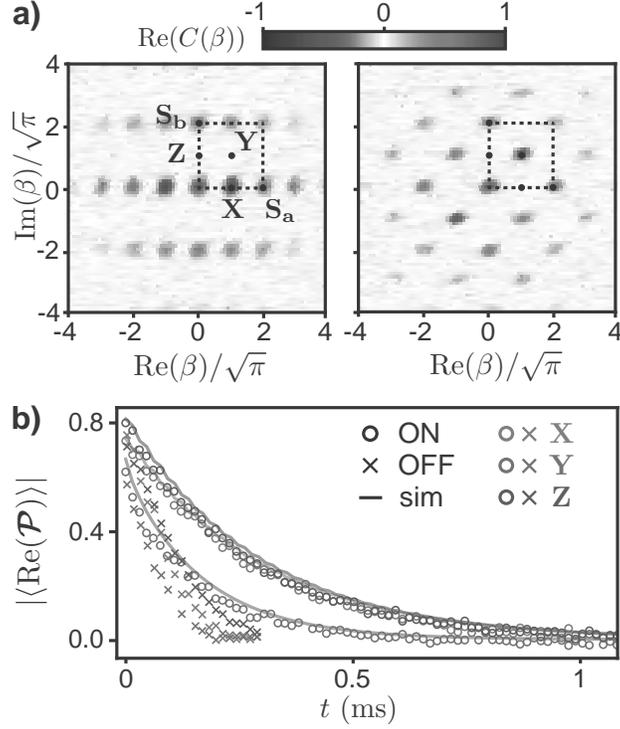


Figure 4: **Initialization and coherence characterization of the logical qubit in the square encoding.** **a)** Left panel: characteristic function of $|-X_L\rangle$ prepared, in steady-state, by applying a feedback \mathbf{Z} -gate conditioned on the outcome $+1$ of a first single-round $\text{Re}(\mathbf{X})$ measurement, before heralding a higher fidelity state on the outcome -1 of a second identical measurement. Right panel: same procedure to prepare $|-Y_L\rangle$. **b)** After preparing $|-X_L\rangle$, $|-Y_L\rangle$ or $|-Z_L\rangle$, the time-decay of the real part of $\mathcal{P} = \mathbf{X}, \mathbf{Y}$ or \mathbf{Z} , respectively, is measured when continuously applying the stabilization sequence (ON) or not (OFF). In the former case, we track deterministic flips of the logical qubit (not visible here) caused by the stabilization protocol (10). The stabilization extends the lifetime of the 3 Bloch vector components to $T_X = T_Z = 275 \mu\text{s}$ and $T_Y = 160 \mu\text{s}$, and the results are quantitatively reproduced by master-equation simulations (lines).

of either \mathbf{X} , \mathbf{Y} or \mathbf{Z} since, for the finite width code we consider, the Clifford states are not strictly ± 1 eigenstates of these operators. Experimentally, after four more rounds of stabilization which approximately project the generated state onto the stabilized code, a second measurement of the same operator yields the same result as the first measurement with probability ~ 0.8 . This imperfect correlation originates equally from the imperfect preparation after the first measurement, and from the finite contrast of the second one. We can further improve the fidelity of preparation to the finitely squeezed Clifford state by substituting in the previous protocol the single measurement of the Pauli operator of the infinitely squeezed code with two consecutive such measurements. In order to prepare $| -X_L \rangle$ (respectively $| -Y_L \rangle$), after a first $\text{Re}(\mathbf{X})$ (respectively $\text{Re}(\mathbf{Y})$) measurement, we apply a feedback displacement $\mathbf{D}(b/2)$ (respectively $\mathbf{D}(a/2)$), which implements a \mathbf{Z} -gate in the ideal code (respectively a \mathbf{X} -gate), if the measurement yields the undesired outcome. We then post-select realizations in which the second measurement yields the desired outcome. The oscillator characteristic function is subsequently measured conditioned on the success of this preparation (see Fig. 4a). The ideal code Pauli operators value in these two cases are respectively $\langle \text{Re}(\mathbf{X}) \rangle = -0.8$ and $\langle \text{Re}(\mathbf{Y}) \rangle = -0.63$. We insist here that these values do not reflect the preparation fidelity to the finitely squeezed logical states $| -X_L \rangle$ and $| -Y_L \rangle$, and the prepared state is as close, within experimental uncertainties, to the target state as allowed by the imperfect code stabilization (see (10) for details). The same methods are applied to prepare other Clifford states (data not shown). In particular, the $| -Z_L \rangle$ state characteristic function is the one of $| -X_L \rangle$ rotated by 90° (not shown).

4 Coherence of the stabilized logical qubit

In order to test the error-correction performance of our stabilization scheme, we prepare one of the logical states $| -X_L \rangle$, $| -Y_L \rangle$ or $| -Z_L \rangle$, then compare the decay of the mean value of the real part of the corresponding operator $\mathcal{P} = \mathbf{X}$, \mathbf{Y} or \mathbf{Z} in time when performing stabilization (open circles in Fig. 4b) or not (crosses). In all three cases, our error-correction protocol extends the coherence of the logical qubit. We extract the stabilized qubit coherence times $T_X = T_Z =$

275 μs and $T_Y = 160 \mu\text{s}$. The shorter coherence time of the \mathbf{Y} Pauli operator, also visible in the unstabilized case, is expected as the distance in phase-space from the probability peaks of the $|+Y_L\rangle$ state to those of the $|-Y_L\rangle$ state is $\sqrt{2}$ shorter than in the case of $|\pm X_L\rangle$ and $|\pm Z_L\rangle$ (see Wigner representation of all Clifford states in (10)). Therefore, diffusive shifts in phase-space induced by photon dissipation cause more flips of the \mathbf{Y} component of the logical qubit Bloch vector. Master-equation simulations again reproduce quantitatively these results (lines). It is worth noting that the lifetime of the \mathbf{X} and \mathbf{Z} components of the stabilized logical qubit are longer than the photon lifetime in the storage oscillator, demonstrating the potential of bosonic quantum error correction.

5 Hexagonal code

It is possible to modify the square GKP code to obtain a more symmetric encoding for which the coherence times of all three Pauli operators are equal. In general, a two-dimensional code manifold can be defined as the +1 eigenspace of two commuting stabilizers $\mathbf{S}_a = \mathbf{D}(a)$ and $\mathbf{S}_b = \mathbf{D}(b)$ with $\text{Im}(a^*b) = 4\pi$. Geometrically, this condition implies that the magnitude of the cross-product of the two vectors representing these stabilizers in characteristic function space span an area of 4π (see Fig. 3b, 5b). In the hexagonal GKP code (9), we choose $b = ae^{i\frac{\pi}{3}}$, which respects the above area condition for $a = \sqrt{\frac{8\pi}{\sqrt{3}}}$. The Pauli operators correspond to equal length displacements $\mathbf{X} = \mathbf{D}(a/2)$, $\mathbf{Y} = \mathbf{D}(b/2)$ and $\mathbf{Z} = \mathbf{D}(c/2)$ with $c = ae^{i\frac{2\pi}{3}}$. For symmetry reasons (10), we also define a third stabilizer $\mathbf{S}_c = \mathbf{Z}^2 = \mathbf{D}(c)$ that commutes with the two others.

We stabilize this code by adapting the protocol described in Sec. 2 (10). Here, measurement of the 3 hexagonal stabilizers followed by small corrective feedback displacements sharpen the peaks from three different directions. These are interleaved with measurement of 3 short displacement operators orthogonal to the stabilizers, followed by feedback displacements of length $a/2$, which trim the envelope. When applying this protocol on the storage mode initialized in

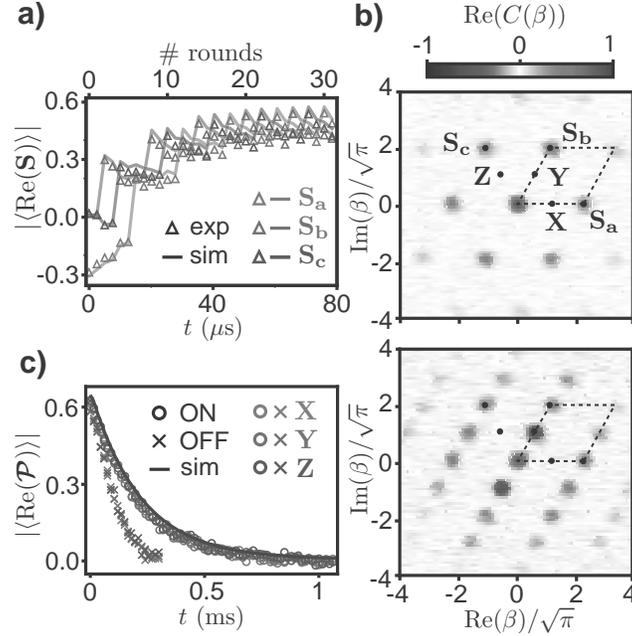


Figure 5: **Stabilization, state-preparation and coherence in the hexagonal code.** **a)** The grid state peaks and envelope are sequentially sharpened and trimmed from three directions. When turning on stabilization from the ground state of the oscillator, the real part of the stabilizers expectation values oscillate every six rounds and increase to rapidly reach a steady-state. **b)** After 200 rounds, the oscillator state is a fully mixed logical state revealing the code structure (top). A Clifford state such as $| - Y_L \rangle$ (bottom) can be prepared by single-round measurement of $\text{Re}(\mathbf{Y})$ followed by a feedback displacement. **c)** Due to the code symmetry, the decay of the logical Bloch vector is isotropic. An exponential fit (black line) indicates a lifetime of 205 μs , enhanced by stabilization.

the ground state, the stabilizers mean values oscillate every 6 rounds as each of these displacement operators is measured in turn, and rapidly converge to a stationary regime for which their value oscillate between 0.4 and 0.55 (see Fig. 5a). We measure the real part of the characteristic function of the fully mixed logical state reached after 200 rounds, which reveals the hexagonal structure of the code (Fig. 5b). Here again, master equation simulations reproduce quantitatively these results and indicate that the generated grid states are characterized by the same squeezing for the peaks as in the square encoding (between 7.5 dB and 9.5 dB in steady-state). Note that the temporary negative value of $\text{Re}(\mathbf{S}_a)$ registered at short times originates from the particular programming of the feedback algorithm on the fast-electronics FPGA board: the oscillator state gets shifted at the beginning of the sequence, which is included in simulations.

As in the square case, we prepare the logical qubit in each Clifford state with a single-round measurement of $\text{Re}(\mathbf{X})$, $\text{Re}(\mathbf{Y})$ or $\text{Re}(\mathbf{Z})$. In Fig. 5b, we show the measured characteristic function of the $| -Y_L \rangle$ state. Note that the characteristic functions of $| -X_L \rangle$ and $| -Z_L \rangle$ are equal to the one of $| -Y_L \rangle$ but rotated by $\pm 60^\circ$ (not shown). Finally we characterize the coherence of the stabilized logical qubit by measuring the decay of the Pauli operator mean values in time. As expected, the decoherence of the logical qubit is now isotropic and significantly extended compared to the unstabilized case, with coherence times of $T_X = T_Y = T_Z = 205 \mu\text{s}$. These values do not allow us to reach break-even (13, 14), but our setup, still in the exploratory stage, is far from having been optimized for this goal.

6 Preparation of an arbitrary logical state

Beyond Clifford states, it is possible to prepare an arbitrary logical state with a simple teleported gate (10, 18). We illustrate this feature by preparing the “magic” state (36)

$|M_L\rangle = \cos(\pi/8)|+X_L\rangle - \sin(\pi/8)|-X_L\rangle$ in both the square and hexagonal codes. The logical qubit is first deterministically prepared in $|+Z_L\rangle$ and the transmon in $|+x\rangle$. We then apply the conditional displacement $\text{CD}(a/2)$ followed by a code-recentering shift $\mathbf{D}(-a/4)$.

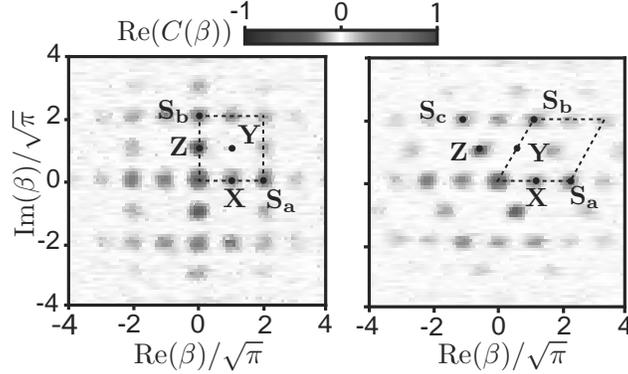


Figure 6: **Magic state preparation** in the square **(a)** and hexagonal **(b)** codes. The sequence $\mathbf{D}(-a/4)\mathbf{CD}(a/2)$ applied on the storage-transmon $|+Z_L\rangle|+x\rangle$ state prepares the entangled Bell state $(|+X_L\rangle|+x\rangle + |-X_L\rangle|-x\rangle)/\sqrt{2}$. Measuring the transmon in any state $|\psi\rangle$ then heralds the preparation of the same state $|\Psi_L\rangle$ in the storage oscillator. Here $|\Psi_L\rangle = |M_L\rangle = \cos(\pi/8)|+X_L\rangle - \sin(\pi/8)|-X_L\rangle$.

This sequence, which is also used to measure **X** (see Sec. 3), prepares the Bell state $(|+X_L\rangle|+x\rangle + |-X_L\rangle|-x\rangle)/\sqrt{2}$. Given any pair of orthogonal logical states $|\pm R_L\rangle$ and denoting as $|\pm r\rangle$ the transmon states with same positions on the Bloch sphere, we can re-write this Bell state as $(|+R_L\rangle|+r\rangle + |-R_L\rangle|-r\rangle)/\sqrt{2}$. Thus, measuring the transmon in $|+r\rangle$ or $|-r\rangle$ heralds the preparation of $|+R_L\rangle$ or $|-R_L\rangle$ in the storage oscillator. We use this protocol to prepare $|M_L\rangle$, whose measured characteristic function is represented in Fig. 6. Note that the preparation can be made deterministic with a simple feedback displacement and the whole sequence can be modified to perform an arbitrary rotation of the logical qubit (10).

7 Logical errors and outlook

The coherence of the logical qubit is limited by two factors. First, the repetition time of the stabilization rounds, despite being a factor of a hundred shorter than the storage mode single-photon lifetime, is not negligible. The transmon dispersive readout and its signal processing using fast electronics accounts for about half of this duration, and the conditional displacement gate for the other half. This gate could in principle be made shorter by using larger displace-

ments of the storage oscillator (amplitude α in Fig. 2c), but in our experiment, this strategy resulted in spurious errors that may be attributed to residual Kerr non-linearity of the oscillator (10). The second factor limiting the logical qubit coherence are transmon errors. Among these, σ_z errors (phase-flips) commute with the storage-transmon interaction Hamiltonian and thus do not propagate to the logical qubit. They are equivalent to transmon readout errors, and are corrected for by subsequent stabilization rounds (10). On the other hand, the σ_x and σ_y transmon errors (bit-flips), as well as excitations to the higher excited states of the transmon, propagate to the logical qubit as they lead to random displacements of the storage mode. Simulations indicate that bit-flips of the transmon and photon loss in the storage mode each account for about half of the logical qubit error rate.

In conclusion, we have demonstrated the successful preparation, readout, and full error-correction of a logical qubit encoded in GKP grid states of a superconducting microwave resonator. Our stabilization scheme only requires a weak dispersive interaction between a harmonic mode of the resonator and an ancillary transmon mode in conjunction with stroboscopic controls of both modes, and significantly enhances the lifetime of *all* logical Pauli operators. This coherence can be further improved by replacing the transmon with a noise-biased ancillary qubit (37, 38). Moreover, in the GKP code based on superconducting cavities and qubits, protected single and multi-qubit Clifford gates can be implemented in a straightforward way, paving the way for the embedding of the logical qubit in a further layer of protection (33–35, 39).

References

1. K. Życzkowski, P. Horodecki, M. Horodecki, R. Horodecki, *Phys. Rev. A* **65**, 012101 (2001).
2. M. A. Nielsen, I. Chuang, *Quantum computation and quantum information* (AAPT, 2002).

3. A. Steane, *Proc. Roy. Soc. London. Series A: Mathematical, Physical and Engineering Sciences* **452**, 2551 (1996).
4. A. G. Fowler, A. M. Stephens, P. Groszkowski, *Phys. Rev. A* **80**, 052312 (2009).
5. C. Nayak, S. H. Simon, A. Stern, M. Freedman, S. D. Sarma, *Rev. Mod. Phys.* **80**, 1083 (2008).
6. P. T. Cochrane, G. J. Milburn, W. J. Munro, *Phys. Rev. A* **59**, 2631 (1999).
7. M. Mirrahimi, *et al.*, *New J. Phys.* **16**, 045014 (2014).
8. Z. Leghtas, *et al.*, *Science* **347**, 853 (2015).
9. D. Gottesman, A. Kitaev, J. Preskill, *Phys. Rev. A* **64**, 012310 (2001).
10. Materials & methods available as supplementary materials on the Science website.
11. Z. Leghtas, *et al.*, *Phys. Rev. Lett.* **111**, 120501 (2013).
12. M. H. Michael, *et al.*, *Phys. Rev. X* **6**, 031006 (2016).
13. N. Ofek, *et al.*, *Nature* **536**, 441 (2016).
14. L. Hu, *et al.*, *Nat. Phys.* **15**, 503 (2019).
15. B. Terhal, D. Weigand, *Phys. Rev. A* **93**, 012315 (2016).
16. V. V. Albert, *et al.*, *Phys. Rev. A* **97**, 032346 (2018).
17. K. Noh, V. V. Albert, L. Jiang, *IEEE Transactions on Information Theory* **65**, 2563 (2019).
18. C. Flühmann, *et al.*, *Nature* **566**, 513 (2019).
19. I. L. Chuang, D. W. Leung, Y. Yamamoto, *Phys. Rev. A* **56**, 1114 (1997).
20. M. Reagor, *et al.*, *Phys. Rev. B* **94**, 014506 (2016).

21. A. Eddins, *et al.*, *Phys. Rev. Lett.* **120**, 040505 (2018).
22. C. Flühmann, V. Negnevitsky, M. Marinelli, J. P. Home, *Phys. Rev. X* **8**, 021001 (2018).
23. D. Riste, J. Van Leeuwen, H.-S. Ku, K. W. Lehnert, L. DiCarlo, *Phys. Rev. Lett.* **109**, 050507 (2012).
24. P. Campagne-Ibarcq, *et al.*, *Phys. Rev. X* **3**, 021008 (2013).
25. A. Y. Kitaev, *arXiv preprint quant-ph/9511026* (1995).
26. K. M. Svore, M. B. Hastings, M. Freedman, *arXiv preprint arXiv:1304.0741* (2013).
27. B. Travaglione, G. J. Milburn, *Phys. Rev. A* **66**, 052322 (2002).
28. S. Pirandola, S. Mancini, D. Vitali, P. Tombesi, *Europhys. Lett.* **68**, 323 (2004).
29. H. M. Vasconcelos, L. Sanz, S. Glancy, *Opt. Lett.* **35**, 3261 (2010).
30. K. R. Motes, B. Q. Baragiola, A. Gilchrist, N. C. Menicucci, *Phys. Rev. A* **95**, 053819 (2017).
31. D. J. Weigand, B. M. Terhal, *Phys. Rev. A* **97**, 022341 (2018).
32. D. F. Walls, G. J. Milburn, *Quantum optics* (Springer Science & Business Media, 2007).
33. S. Glancy, E. Knill, *Phys. Rev. A* **73**, 012325 (2006).
34. K. Fukui, A. Tomita, A. Okamoto, K. Fujii, *Phys. Rev. X* **8**, 021054 (2018).
35. C. Vuillot, H. Asasi, Y. Wang, L. P. Pryadko, B. M. Terhal, *Phys. Rev. A* **99**, 032344 (2019).
36. S. Bravyi, A. Kitaev, *Phys. Rev. A* **71**, 022316 (2005).
37. S. Puri, *et al.*, *arXiv preprint arXiv:1807.09334* (2018).
38. Y. Shi, C. Chamberland, A. W. Cross, *arXiv preprint arXiv:1905.00903* (2019).

39. B. Q. Baragiola, G. Pantaleoni, R. N. Alexander, A. Karanjai, N. C. Menicucci, *arXiv preprint arXiv:1903.00012* (2019).
40. N. E. Frattini, V. V. Sivak, A. Lingenfelter, S. Shankar, M. H. Devoret, *Phys. Rev. A* **10**, 054020 (2018).
41. D. T. McClure, *et al.*, *Phys. Rev. A* **5**, 011001 (2016).
42. S. E. Nigg, *et al.*, *Phys. Rev. Lett.* **108**, 240502 (2012).
43. J. M. Chow, *et al.*, *Phys. Rev. A* **82**, 040305 (2010).
44. J. R. Johansson, P. D. Nation, F. Nori, *Comp. Phys. Com.* **184**, 1234 (2013).

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Supplementary materials

S1 Experimental design

The experimental system is schematically represented in Fig. S1, and its parameters are summarized on Table 1. The storage and readout oscillators are the fundamental modes of two rectangular coaxial microwave cavities machined out of a single 6061 aluminum alloy block. A section of rectangular waveguide with the same dimensions extends each cavity and is closed off by a top lid coated with copper powder mixed in *Apiezon Wax W* (brown layer in Fig. S1). We embed in the wax a copper braid attached to the base-plate of the refrigerator in order to thermally anchor this non-magnetic, highly dissipative material, which damps and thermalizes the higher frequency modes of the structure. The cavities are protected from external magnetic fields using an *Amumetal A4K* shield.

The cavities are bridged by a single transmon superconducting circuit, made of a double-angle-evaporated Al/AlO_x/Al Josephson junction (inductance 7.3 nH) bridging two 0.7 mm-by-0.4 mm rectangular aluminum pads. It is fabricated using the bridge-free fabrication technique on a double-side-polished 5 mm-by-37.5 mm chip of c-plane sapphire with a 0.43 mm thickness. The chip is clamped from two sides (out of the representation plane in Fig. S1) by compressing it between thermally anchored copper blocks covered with a 200 μm thick indium foil.

The full wiring diagram of the experiment is depicted in Fig. S1. The microwave lines are filtered using both homemade *eccosorb*-based dissipative filters and commercial reflective *K&L* filters. The pulses used to probe and control the system are generated at room temperature by IQ mixing of a local oscillator (LO) provided by a microwave source at $\omega + \omega_h$ with a low frequency signal at ω_h ($50 \text{ MHz} < \omega_h/2\pi < 100 \text{ MHz}$) with arbitrary envelope delivered by the Digital to Analog Converter (DAC) of an integrated FPGA system. For reading out the transmon, the

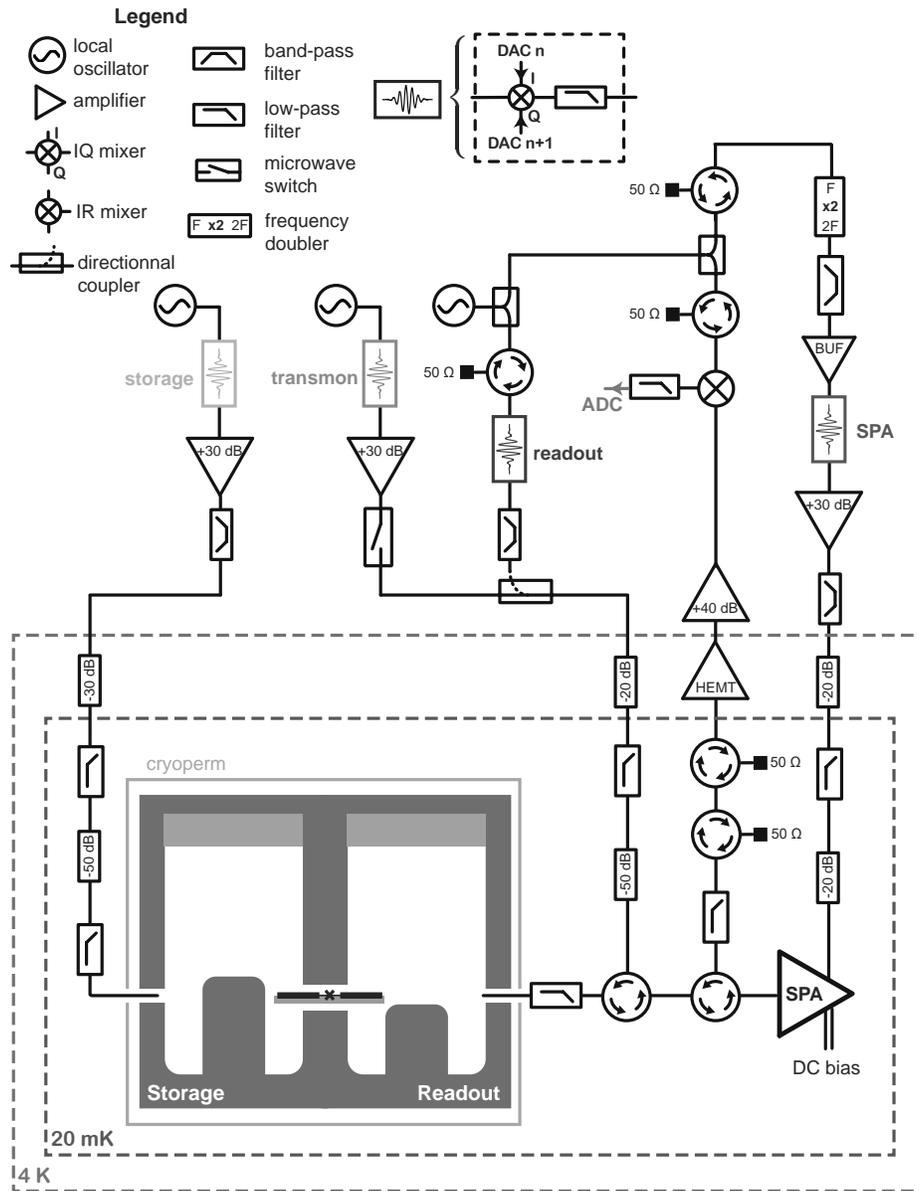


Figure S1: **Wiring diagram.** Two coaxial microwave resonators (gray) bridged by a single transmon superconducting qubit (black) fabricated on a sapphire chip (blue, see text for details) are anchored on a dilution refrigerator base plate. The cavities are machined out of a single aluminum block, and closed by a lid coated with copper powder mixed in *Apiezon Wax W* (brown). Microwave pulses used to probe or control the system are generated by IQ mixing of a local oscillator (LO) with a low frequency signal with arbitrary envelope delivered by the DAC of an integrated FPGA system. The microwaves propagate down heavily attenuated lines to the system. Circulators are used to route the reflected readout signal to a near quantum limited SNAIL-parametric amplifier (SPA) (40) used in phase-sensitive mode (pulsed pump microwave at twice the readout signal frequency). The signal is further amplified and down-converted at room-temperature before digitization by the FPGA board.

Storage oscillator single-photon lifetime	T_s	245 μs
Storage oscillator frequency	$\omega_s/2\pi$	5.26 GHz
Storage oscillator Kerr anharmonicity	$K_s/2\pi$	1 Hz
Transmon energy lifetime	T_1	50 μs
Transmon coherence lifetime (echo)	T_{2e}	60 μs
Transmon coherence lifetime (Ramsey)	T_{2R}	8 μs
Transmon resonance frequency	$\omega_t/2\pi$	6.71 GHz
Transmon anharmonicity	$K_t/2\pi$	193 MHz
Readout oscillator single-photon lifetime	T_r	65 ns
Readout oscillator frequency	$\omega_r/2\pi$	8.2 GHz
Storage-transmon dispersive shift	$\chi/2\pi$	28 kHz
Readout-transmon dispersive shift	$\chi_r/2\pi$	1 MHz
Jump rate to higher transmon levels during stabilization	$\Gamma_{\rightarrow f}$	$(3 \text{ ms})^{-1}$

Table 1: **Measured experimental parameters.** Parameters entering the master-equation simulations described on Sec. S2.6 are highlighted in gray.

signal reflected off from the cavity is pre-amplified by reflection on a near quantum limited SNAIL-parametric amplifier (SPA) (40) used in phase-sensitive mode: a pulsed, high-power, pump at twice the readout signal frequency is generated by frequency-doubling and IQ-mixing the signal from the same LO. The readout signal is then further amplified, down-converted at room-temperature and digitized by the FPGA system.

S2 Experimental parameters characterization

S2.1 Transmon readout

The transmon is readout using a dedicated resonator (see Fig. S1) overcoupled to an output line (photon exit rate $\kappa_r = 2\pi \times 2.5 \text{ MHz}$) forming a near-quantum limited amplification chain. The transmon and the readout oscillator are dispersively coupled with a dispersive shift $\chi_r = 2\pi \times 1 \text{ MHz}$. We measure the transmon with the largest possible photon number without degrading its T_1 . Note that the dispersive coupling between the readout mode and the storage mode $\chi_{sr} = \frac{\chi\chi_r}{2K_t} \sim 2\pi \times 70 \text{ Hz}$ (see Table. 1) is small enough that the storage mode is not dephased by this measurement.

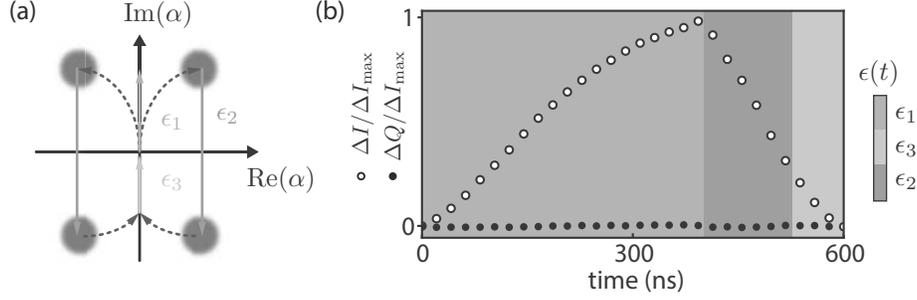


Figure S2: **Fast transmon readout** **a)** Trajectories of the readout oscillator state in phase-space conditioned on the transmon state. By alternating positive and negative amplitude drives of the oscillator, the two trajectories separate then recombine (see text). **b)** Measured differential signal when the transmon is prepared in $|\pm z\rangle$. The reflected signal from the readout resonator is amplified (near quantum limited amplifying chain including a phase-sensitive parametric amplifier (40)). The down-converted, digitized and window-integrated signal quadratures I and Q reflect the intra-cavity field coordinates $\text{Re}(\alpha)$ and $\text{Im}(\alpha)$, respectively amplified and de-amplified.

In order to shorten the ringdown time of the resonator, which is desired to limit the dead time before manipulating the transmon again, we use a fast unloading protocol (41). This modified dispersive readout scheme is schematized in Fig. S2a. It starts by driving the resonator with a positive amplitude ϵ_1 . The readout oscillator state gets displaced and separates along an angle given by the state of the transmon. We then revert the amplitude of the drive to a large negative value ϵ_2 (green arrows) to displace the two possible coherent states to the opposite direction in phase-space. The two states then recombine and are brought back to the origin using a third drive with a positive amplitude ϵ_3 . One can understand this sequence by comparing it to the conditional displacement sequence described in Sec. 1. The joint transmon-readout Hamiltonian is similar to the transmon-storage one given in Eq. (2). However, here, we do not flip the transmon state in the middle of the sequence. Thus, the overall conditional displacement resulting from the action of the second term in Eq. (2) is canceled: depending on the transmon state, the oscillator trajectories in phase-space first separate, then recombine. Integrating the

transient signal yields the transmon state, with the oscillator left in the ground state at the end of the sequence.

A noticeable difference with the case of the storage mode conditional displacement is that the photon lifetime in the oscillator is, here, of the same order as the sequence duration. We optimize the exact amplitudes and phases of $\epsilon_{1,2,3}$ using an algorithm that minimizes the number of photons in the oscillator after 600 ns. As a result, the transmon can be rotated 100 ns later (see Fig. 2b) with no detectable spurious dephasing. The measured differential signal for the transmon prepared in $|\pm z\rangle$ is plotted on Fig. S2b. From its maximum value, the signal drops by more than 99 % in about 200 ns, which is shorter than the time $\sim 10T_R = 650$ ns required for the oscillator field amplitude to decay by a comparable amount with passive photon damping. The overall readout fidelity is close to unity, and the sequence is modeled as a perfect instantaneous projection of the transmon state after 350 ns in master-equation simulations (see Sec. S2.6).

S2.2 Displacement length characterization

The transmission of the input lines cannot be precisely known, so the storage oscillator displacement rate for a given room-temperature pulse power is *a priori* unknown. We thus need to calibrate the displacement $\mathbf{D}(\delta)$ associated with a given driving pulse amplitude (the pulses used for feedback displacements have a Gaussian temporal shape with a fixed standard deviation of 5 ns). Similarly, for the driving sequence represented in Fig. 2b, we need to estimate the conditional displacement $\mathbf{CD}(\beta)$ corresponding to a given drive amplitude. Note that we could in principle deduce one of these scaling factors from the other (conditional displacements result from large unconditional ones), but such an estimate would be imprecise due to the vastly different timescales involved.

We start by estimating the conditional displacements scale. This is done by measuring the characteristic function of the vacuum state, which reads $C_{\text{vac}}(\beta) = e^{-\frac{|\beta|^2}{4}}$ with our convention.

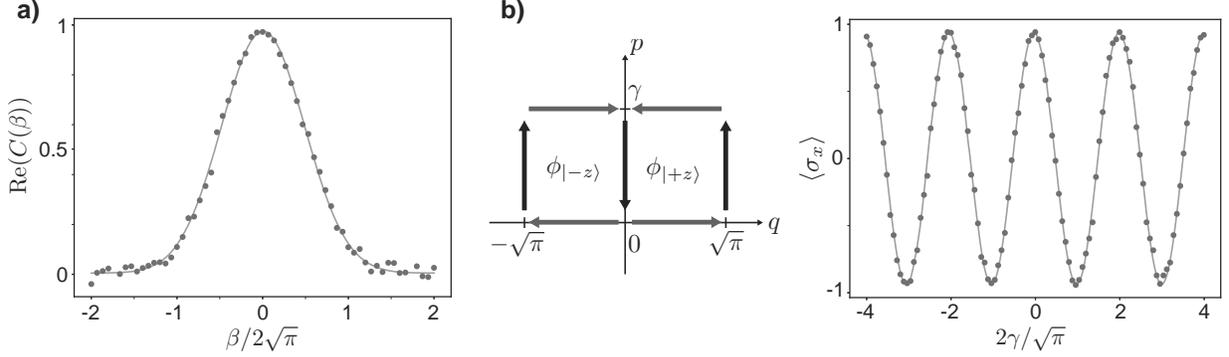


Figure S3: **Displacement length calibration.** **a)** We adjust the scaling of the conditional displacements so that the measured characteristic function of the vacuum state (blue circles) is a Gaussian with standard deviation $\sqrt{2}$ (orange line). **b)** Unconditional displacements are calibrated by applying a sequence of conditional displacements $\text{CD}(2\sqrt{\pi})$ (red arrows) and unconditional ones $\mathbf{D}(\pm\gamma)$ (black arrows) on the transmon prepared in $|+x\rangle$. At the end of the sequence, the transmon state has picked up a geometrical phase $\phi = -\phi_{|+z\rangle} + \phi_{|-z\rangle}$ where $\phi_{|\pm z\rangle}$ is the oriented area circled in the oscillator phase-space when the transmon is in $|\pm z\rangle$. The displacement scaling is adjusted so that varying γ results in oscillations of $\langle\sigma_x\rangle$ (blue circles) with a period $\sqrt{\pi}$ (orange line).

We assume the oscillator to be in the vacuum state at equilibrium and, as described in Sec. 1, we measure $C(\beta)$ (plotted in Fig. S3a) by averaging a transmon measurement following a conditional displacement $\text{CD}(\beta)$. Neglecting the transmon bit-flip errors during the conditional displacement and spurious thermal excitations of the oscillator, the scale of the conditional displacements is then set by adjusting $C(\beta)$ to a Gaussian with standard deviation $\sqrt{2}$. As for the other parameters relevant to the performance of our stabilization protocol, this original calibration of the conditional displacements is finely tuned (variation of about 1 %) by empirically maximizing the coherence of the logical qubit under stabilization.

We now turn to calibrating the unconditional displacements length. This is done by performing the sequence of gates $\mathbf{D}(-i\gamma)\text{CD}(-2\sqrt{\pi})\mathbf{D}(i\gamma)\text{CD}(2\sqrt{\pi})$ on the oscillator initially in vacuum and the transmon in $|+x\rangle$. γ is the *a priori* unknown displacement length we want to calibrate. As represented in Fig. S3b, the storage oscillator state follows a transmon-dependent

trajectory in phase space. At the end of the sequence, the two possible trajectories recombine, so that the oscillator and the transmon are disentangled. However, the two trajectories circle around two opposed oriented areas in phase-space, so that the transmon picks up a geometrical phase: at the end of the sequence, its state reads $e^{i\sqrt{\pi}\gamma}|+z\rangle + e^{-i\sqrt{\pi}\gamma}|-z\rangle$ up to normalization. When varying γ , a final σ_x measurement of the transmon yields oscillations with a period $\sqrt{\pi}$. The recorded oscillations plotted in Fig. S3c are used to calibrate the displacement length.

S2.3 Single-photon lifetime in the storage mode

We estimate the photon lifetime in the storage mode by recording the decay of a coherent state amplitude and neglecting any pure dephasing (see Sec. S2.7 for comments on this hypothesis). We start by displacing the oscillator state by δ_0 (coherent state $|\delta_0/\sqrt{2}\rangle$ with our notation), the transmon being in its ground state. The free decay of the coherent field is expected to be

$$\delta(t) = \delta_0 e^{-i\Delta_s t - \frac{t}{2T_s}}. \quad (\text{S1})$$

where Δ_s is the detuning between the microwave used to displace the oscillator (loading and characteristic function measurement) and the resonance frequency. The characteristic function of the oscillator in this coherent state reads

$$C(\beta, t) = e^{-\frac{1}{4}|\beta|^2} e^{-i\text{Im}(\delta(t)\beta^*)}. \quad (\text{S2})$$

We measure $C(\beta, t)$ for $\beta \in \mathbb{R}$ and $\beta \in i\mathbb{R}$ and fit this signal to obtain $\delta(t)$. The fitted value as a function of time is represented in Fig. S4a. The coherent state rotates in phase-space and its amplitude decays exponentially with a characteristic time $2T_s = 490 \mu\text{s}$.

S2.4 Storage mode resonance frequency and dispersive coupling to the transmon

From the same field amplitude decay measurement described in the previous section, we can extract the storage mode resonance frequency $\omega_s - \frac{\chi}{2}$ when the transmon is in its ground state (see Eq. 1). We repeat this measurement after preparing the transmon in its excited state, in

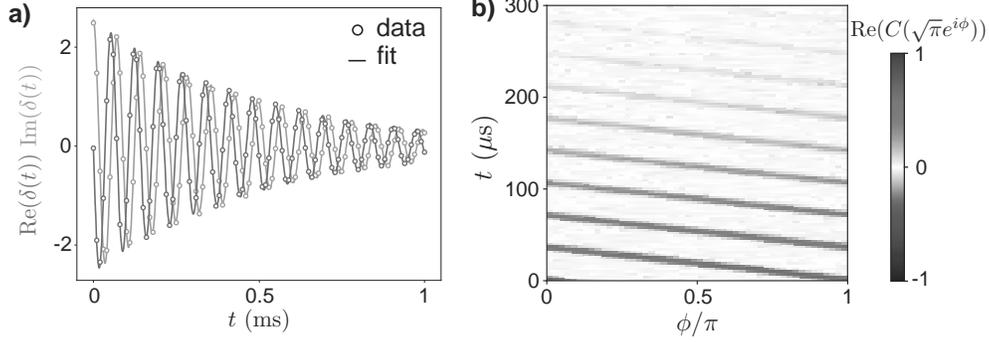


Figure S4: **Storage oscillator characterization** **a)** With the transmon in its ground state, the free decay of a coherent state $|\delta(t)/\sqrt{2}\rangle$ in the storage oscillator is monitored by measuring the characteristic function at a varying time t and fitting it using Eq. (S2). The state amplitude decays exponentially with characteristic time $2T_s = 490 \mu\text{s}$. **b)** After preparing the logical qubit in $|+X_L\rangle$, the free decay of the \mathbf{X} component of the logical Bloch vector (stabilization turned off) is monitored by measuring circular cuts of the real part of the characteristic function $C(\beta)$ with $|\beta| = a/2$. The function maximum is $\langle \text{Re}(\mathbf{X}) \rangle$, and, the transmon being in its ground state, its position rotates at $-\chi/2$.

which case the mode resonates at $\omega_s + \frac{\chi}{2}$. We thus get a first estimate of the oscillator frequency ω_s (the rotating frame frequency of the Hamiltonian (2)), and of the dispersive shift χ .

However, transmon relaxation events during the field decay and during the characteristic function measurement sequence limit the precision of this calibration. A more precise estimate of ω_s within ± 100 Hz is obtained by varying the frequency of all the oscillator control pulses during stabilization and empirically maximizing the logical qubit coherence time. χ is also more finely estimated by considering the decay of the logical Pauli operators expectation value when the stabilization is OFF. For a varying time t after preparing the logical qubit in $|+X_L\rangle$ and resetting the transmon in its ground state, we measure circular cuts of the characteristic function $\text{Re}(C(\frac{a}{2}e^{i\phi}))$ with $\phi \in [0, \pi]$ in Fig. S4b. At $t = 0$, it takes its maximal value, corresponding to $\langle \text{Re}(\mathbf{X}) \rangle$, at $\phi = 0$. For $t > 0$, the position of the maximum rotates at $-\frac{\chi}{2}$, which is the detuning between the working frequency and the oscillator frequency for the transmon in its ground state. We thus get $\chi = 2\pi \times 28$ kHz, which is the value used to estimate the maximum pho-

ton number of $|\alpha^{\max}|^2 = 320$ reached when performing the conditional displacements for the measurement of the stabilizers. We can also estimate the storage oscillator Kerr anharmonicity inherited from its hybridization to the transmon mode (42) to be $K_s = \frac{\chi^2}{4K_t} = 2\pi \times 1 \text{ Hz}$, where $K_t = 2\pi \times 193 \text{ MHz}$ is the transmon anharmonicity.

Extracting the maximum value of the circular characteristic function cuts at all times after preparing $|+X_L\rangle$, $|+Y_L\rangle$ and $|+Z_L\rangle$, we reconstruct the decay of the three components of the logical Bloch vector $\langle \text{Re}(\mathbf{X}) \rangle_{\text{OFF}}$, $\langle \text{Re}(\mathbf{Y}) \rangle_{\text{OFF}}$ and $\langle \text{Re}(\mathbf{Z}) \rangle_{\text{OFF}}$, used as a baseline to evaluate the quantum error-correction performances of our stabilization scheme (color crosses in Fig. 4b and Fig. 5c).

S2.5 Excitation to higher levels of the transmon

Excitations of the transmon to its second excited state $|f\rangle$ lead to a depolarization of the logical qubit. Indeed, while our protocol includes resets of the transmon every stabilization round by non-demolition readout and feedback, the control pulses on the $|g\rangle \leftrightarrow |e\rangle$ transition are ineffective when the transmon is in $|f\rangle$. Thus, when the transmon excites to $|f\rangle$, the oscillator state rotates at $\Delta_s = -\frac{3}{2}\chi$, which is the detuning between the working frequency and the dispersively shifted oscillator frequency, until the transmon spontaneously relaxes back to $|e\rangle$. Such relaxation happens after a typical time $T_1/2$, which is longer than $1/\Delta_s$, so the oscillator state is rotated by a random angle during the time the transmon spends in $|f\rangle$. Once the transmon is back in the $\{|g\rangle, |e\rangle\}$ manifold, the stabilization turns back on, projecting the oscillator state in the code manifold. However, the logical qubit ends up in a random (fully mixed) state.

Such excitations of the transmon can be either thermally activated (the equilibrium occupation of the first excited state of the transmon is found to be $\sim 1\%$) or triggered by the fast control pulses applied on the $|g\rangle \leftrightarrow |e\rangle$ transition. Indeed, the total duration of these pulses is 30 ns, which corresponds to a spectral width of the same order as the transmon anharmonicity.

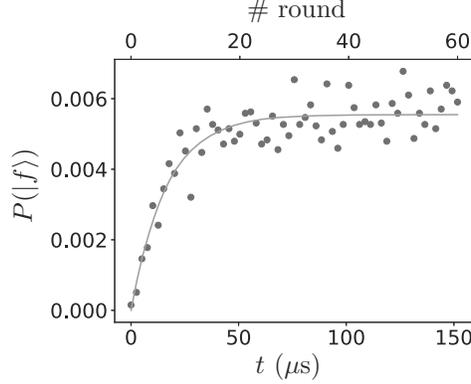


Figure S5: **Excitation to $|f\rangle$.** From the thermal equilibrium state of the transmon (negligible occupation of the $|f\rangle$ level), the same pulse sequence used for stabilizing the GKP code manifold is applied to the system. The occupation of $|f\rangle$ is measured as a function of round number using a modified dispersive readout. The finite readout fidelity $F \approx 0.9$ is calibrated and corrected for. The orange line is an exponential fit with characteristic time $T = 16.5 \mu\text{s} \lesssim T_1/2$.

We limit these undesired excitations by using derivative removal via adiabatic gate (DRAG) pulse shaping (43).

We estimate the rate $\Gamma_{\rightarrow f}$ of transmon excitation to $|f\rangle$ —translating into a spurious depolarization of the logical qubit with identical rate—by measuring the $|f\rangle$ level occupation as a function of round number during stabilization (see Fig. S5), and using a hidden Markov model. Before the stabilization is turned on, the probability of occupation of $|f\rangle$ is negligible (thermal equilibrium). When the stabilization is on, the average occupation of the first excited state instantaneously becomes $P(|e\rangle) = 0.5$, and the $|f\rangle$ level starts to fill. $P(|f\rangle)$ reaches a new equilibrium $P_{eq}(|f\rangle) = P(|e\rangle) \frac{\Gamma_{ef}}{\Gamma_{fe}}$ with a characteristic time $1/(\Gamma_{ef}/2 + \Gamma_{fe})$, where Γ_{ef} (respectively Γ_{fe}) is the excitation rate to $|f\rangle$ (respectively de-excitation rate to $|e\rangle$) when the transmon is in $|e\rangle$ (respectively $|f\rangle$). Fitting $P(|f\rangle)$ with an exponential, we extract Γ_{ef} and then get $\Gamma_{\rightarrow f} = P(|e\rangle)\Gamma_{ef} = (3 \text{ ms})^{-1}$.

S2.6 Master equation simulations

In this section, we briefly describe the master-equation simulations reproducing all experimental data presented in the main text. The parameters entering the simulations are summarized in Tab. I and are independently calibrated.

The joint storage-transmon state is represented by a 300×300 density matrix (150×2 -dimension Hilbert space). Its dynamics is found by solving the Lindblad master-equation

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[\tilde{\mathbf{H}}, \rho] + \frac{1}{T_s}\mathcal{D}[\mathbf{a}]\rho + \frac{1}{T_1}\mathcal{D}[\sigma_-]\rho + \frac{1}{2T_\phi}\mathcal{D}[\sigma_z]\rho, \quad (\text{S3})$$

where $\mathcal{D}[\mathbf{L}]\rho = \mathbf{L}\rho\mathbf{L}^\dagger - \frac{1}{2}(\mathbf{L}^\dagger\mathbf{L}\rho + \rho\mathbf{L}^\dagger\mathbf{L})$ and $T_\phi = 1/(1/T_2 - 1/2T_1)$ is the transmon pure dephasing time. The Hamiltonian $\tilde{\mathbf{H}}$ is taken in the displaced frame during the conditional displacements and following Eq. (2), it reads

$$\frac{\tilde{\mathbf{H}}}{\hbar} = -\frac{\chi}{2}(\mathbf{a}^\dagger + \alpha^*)(\mathbf{a} + \alpha)\sigma_z - \frac{K_s}{2}\left((\mathbf{a}^\dagger + \alpha^*)(\mathbf{a} + \alpha)\right)^2, \quad (\text{S4})$$

where we have included for completeness the Kerr anharmonicity inherited by the oscillator.

We numerically solve this equation using *Qutip* (44) during each round of stabilization. The transmon control pulses and the feedback displacements of the storage oscillator are modeled as instantaneous unitary evolutions (rotation and displacement operators applied on the density matrix). The transmon readout is modeled as a perfect, instantaneous projection taking place at the middle of the actual readout pulse (time t_1). We compute the two possible un-normalized density matrices resulting from this measurement $\rho_\pm(t_1) = \mathbf{M}_\pm\rho(t_1)\mathbf{M}_\pm^\dagger$ corresponding to the two possible measurement outcomes, with $\mathbf{M}_\pm = |\pm z\rangle\langle\pm z|$. We simulate separately the evolution of these two matrices. After a time t_2 including the second half of the readout pulse and the delay required for the fast-electronics board to process the measurement signal, we apply the feedback operations \mathbf{U}_\pm (oscillator displacement and rotation of the transmon) corresponding to each measurement outcome and sum the two matrices

$$\rho(t_1 + t_2) = \mathbf{U}_+ \rho_+(t_1 + t_2) \mathbf{U}_+^\dagger + \mathbf{U}_- \rho_-(t_1 + t_2) \mathbf{U}_-^\dagger.$$

These simulations allow us to reproduce quantitatively the expectation values of the stabilizers measured experimentally (see Figs. 3a-5a and Fig. 5a) as well as the preparation fidelity of all logical states. Occupation of the $|f\rangle$ level of the transmon (below 1 %) is then neglected. On the other hand, when considering the lifetime of the components of the logical qubit Bloch vector, the depolarization induced by transmon excitations to $|f\rangle$ with rate $\Gamma_{\rightarrow f}$ (see Sec. S2.5) cannot be neglected: the decaying coherence signal returned by simulations is fitted with an exponential law, and the supplementary dephasing rate $\Gamma_{\rightarrow f}$ is added to the simulated decay rate to get the predicted decay plotted in Figs. 4b-5c.

Overall, the simulations reproduce experimental data within a $\sim 2\%$ margin, except for the expectation values of the hexagonal code stabilizers, for which the mismatch increases to 5 %. This discrepancy is probably explained by the fact that this particular data set was recorded several days later than the rest of the experimental results presented in the paper, without re-tuning the experimental setup parameters (in particular compensating for drifts in the various pulses power).

We also use master equation simulations to estimate the impact of the various error channels on the error-correction performances. The results of these simulations are summarized on Table 2. We do not give a quantitative error budget as some errors appear to slightly compound each other and their effect cannot be quantitatively estimated independently. Qualitatively, propagation of transmon bit-flips errors (see Sec. S6.2) and errors linked to the intrinsic storage photon dissipation (see Sec. S4.1) dominate the error budget and each account for about half of the logical qubit errors. As expected, phase-flips of the transmon only marginally affect the stabilization performances (see Sec. S6.1).

Error channel	X,Z square code		Y square code		X,Y,Z hex. code	
	T (μs)	$\Gamma/2\pi$ (kHz)	T (μs)	$\Gamma/2\pi$ (kHz)	T (μs)	$\Gamma/2\pi$ (kHz)
Dissip.	890	0.18	450	0.36	600	0.27
Bit-flips	650	0.24	340	0.47	470	0.34
Phase-flips	12000	0.013	6500	0.025	8000	0.020
Dissip. & Bit-flips	340	0.47	180	0.88	250	0.64
Leakage to $ f\rangle$	3000	0.053	3000	0.053	3000	0.053
All	295	0.54	165	0.96	220	0.72

Table 2: **Impact of the various error channels on error-correction performances.** We use master equation simulations to estimate the lifetime of the logical qubit Bloch vector components in presence of each noise process separately (storage mode photon dissipation, bit-flips and phase-flips of the transmon and excitation of the transmon to the $|f\rangle$ level). The corresponding decay rate is given in each case. The storage oscillator photon dissipation and the transmon bit-flips dominate the error budget and slightly compound each other.

S2.7 Non-linearity and pure dephasing of the storage mode

A crucial feature of our experimental system is the low value of the storage oscillator non-linearity inherited from its hybridization to the transmon mode. As mentioned in Sec. S2.4, residual Kerr non-linearity is estimated to be $K_s \approx 2\pi \times 1$ Hz. A larger value could limit the coherence of the logical qubit in two ways. First, Kerr non-linearity distorts the grid states, which can be understood at first order as a position and momentum dependent rotation term in the oscillator phase-space. This effect gets stronger as the grid state envelope gets broader, which makes the envelope-trimming procedure described in Sec. 2 even more relevant. Second, spurious resonant terms appear in the displaced frame-Hamiltonian given in Eq. (S4). These terms of order $K_s|\alpha|^2$ lead to distortion of the grid state when performing too fast conditional displacements. In the experiment, we used a maximum value $|\alpha_{\max}|^2 = 320$ photons. For twice faster conditional displacements (more than 4 times larger maximum photon number), we observed a lower logical qubit coherence time, which is reproduced by simulations. Note that overall, designing the experiment with a lower transmon-storage dispersive coupling should decrease the oscillator Kerr non-linearity while retaining the ability to perform conditional displacements with the same speed: the conditional displacement rate scales as $\chi|\alpha|$ while the

spurious non-linear terms appear at a given value of $\chi^2|\alpha|^2$.

One can transpose this reasoning to the case of higher order non linearities and decoherence processes of the resonator. In particular, we can place an upper bound on the storage mode pure dephasing rate $\kappa_\phi \lesssim 2\pi \times 1$ Hz. Indeed, adding the decoherence operator $\kappa_\phi \mathcal{D}[\mathbf{a}^\dagger \mathbf{a}] \rho$ in the Lindblad master equation also results in spurious terms of order $|\alpha|^2$ when moving to the displaced frame, and simulations indicate that a larger pure dephasing rate would result in a shorter logical qubit coherence time than reported in the experiment. This upper bound justifies our approach for calibrating the storage oscillator energy relaxation time as described in Sec. S2.3.

S3 The ideal GKP code

S3.1 Square code

The two-dimensional square GKP code is stabilized by the two operators $\mathbf{S}_a = \mathbf{D}(a = 2\sqrt{\pi})$ and $\mathbf{S}_b = \mathbf{D}(b = ia)$ (9). One can geometrically verify that these displacement operators commute using $\mathbf{D}(-\beta)\mathbf{D}(-\alpha)\mathbf{D}(\beta)\mathbf{D}(\alpha) = e^{i\mathcal{A}}$, where $\mathcal{A} = \text{Im}(\alpha^*\beta)$ is the oriented surface circled in phase-space when applying sequentially the four displacement operators. In Fig. S6, we represent the Wigner quasi-probability distributions of all 6 Clifford states as well as the maximally mixed logical state. The code words are +1 eigenstates of the stabilizers, which can be viewed as displacement operators by a along $q = r_0$ and $p = r_{\pi/2}$, so that all the Wigner functions are a -periodic in r_0 and $r_{\pi/2}$ (we use the notation r_θ for the generalized phase-space coordinate along the axis forming an angle θ with the q -axis). Alternatively, one can view the stabilizer $\mathbf{S}_a = e^{-ia\mathbf{x}_{\pi/2}}$ (respectively $\mathbf{S}_b = e^{-ia\mathbf{x}_\pi}$) as a function of $\mathbf{r}_{\pi/2}$ (respectively \mathbf{r}_π) of modulus 1. Thus, for all the ideal logical states, the marginal probability distributions $P_\theta = \int_{r_{\theta+\pi/2}} W(r_\theta, r_{\theta+\pi/2}) dr_{\theta+\pi/2}$ along the directions of the reciprocal lattice $\theta = \pi/2$ and $\theta = \pi$ (black lines in Fig. S6) have a support limited to the antecedents of +1 by this function. This function is $(2\pi)/a$ -periodic and these antecedents verify $r_\theta = 0 \pmod{2\pi/a}$.

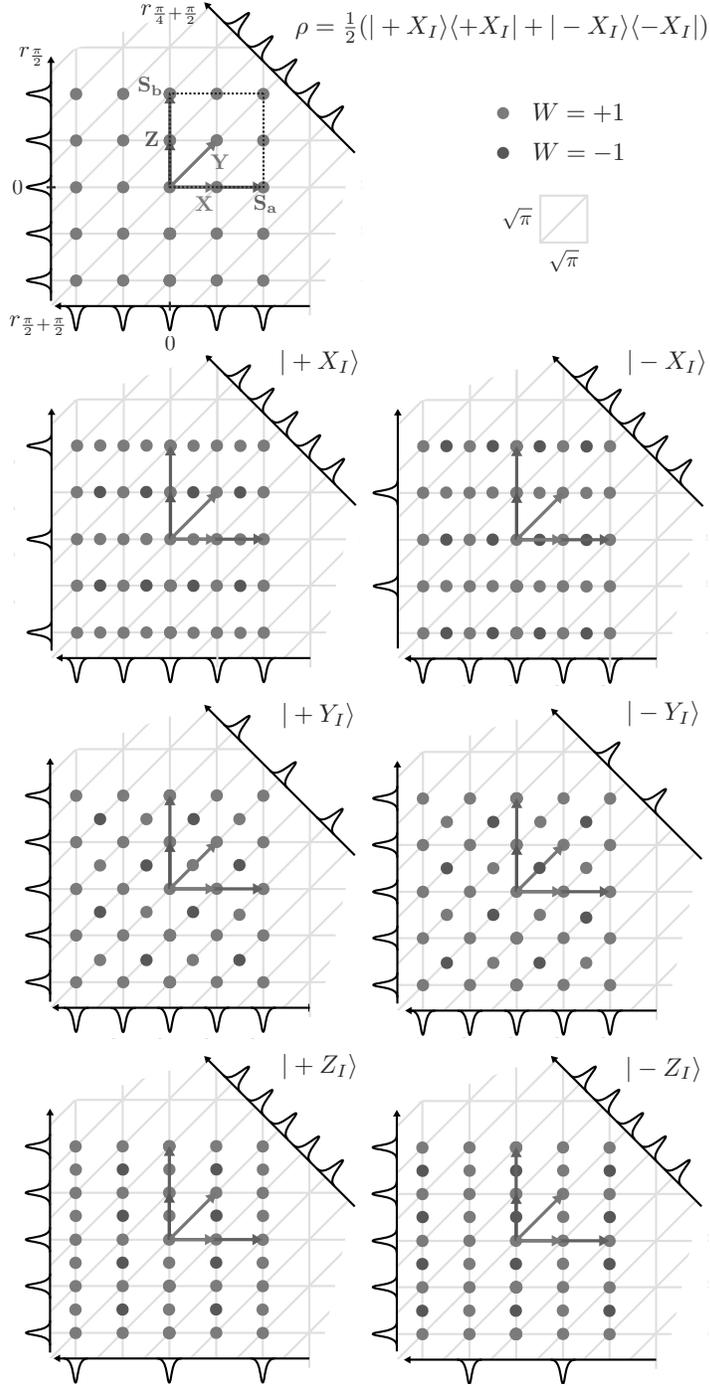


Figure S6: **Logical states in the ideal square code.** Wigner quasi-probability distributions and their marginals along axes orthogonal to a , b , and $a + b$ are schematically represented for the maximally mixed logical state—revealing the square lattice structure, highlighted with gray lines—and the 6 Clifford states. We give quasi-probability peaks (red and blue dots) as well as peaks of the marginals (black lines) a finite extension for better visualization. The distribution extends infinitely in phase-space and is obtained by tessellating the pattern of the unit cell delimited by a black dashed line (area 4π).

The logical Pauli operators are $\mathbf{X} = \mathbf{D}(a/2)$, $\mathbf{Y} = \mathbf{D}((a + b)/2)$ and $\mathbf{Z} = \mathbf{D}(b/2)$. They commute with the stabilizers and verify the Pauli group composition rules inside the code. Following similar arguments as for the stabilizers, we show that the Wigner functions of the states $|\pm X_I\rangle$, $|\pm Y_I\rangle$ and $|\pm Z_I\rangle$ are $a/2$ -periodic along r_θ with respectively $\theta = 0, \pi/4, \pi/2$, and one peak out of two vanishes for their probability distribution $P_{\theta+\pi/2}$ along the reciprocal axis. In particular, we notice that the probability peaks of $|+Y_I\rangle$ are $\sqrt{2}$ closer to those of $|-Y_I\rangle$ than for $|\pm X_I\rangle$ or $|\pm Z_I\rangle$, which explains the shorter lifetime of the \mathbf{Y} component of the logical Bloch vector.

Finally, let us note that the non-zero values of the Wigner functions are points on a square lattice with cell side twice smaller than that of the lattice defining the actual probability peaks of $P_{\pi/2}$ and P_π (highlighted with gray lines in Fig. S6). Thus, by analogy with Schrödinger cat states (32), we can interpret the non-zero values of the Wigner function on the larger lattice as “probability blobs”, when those shifted by half the lattice period are “interference fringes”.

S3.2 Hexagonal code

The two-dimensional hexagonal GKP code is stabilized by the two commuting displacement operators $\mathbf{S}_a = \mathbf{D}(a = \sqrt{\frac{8\pi}{\sqrt{3}}})$ and $\mathbf{S}_b = \mathbf{D}(b = ae^{i\frac{\pi}{3}})$ (9). We also consider a third displacement operator, which commutes with the two previous ones, $\mathbf{S}_c = \mathbf{D}(c = ae^{i\frac{2\pi}{3}})$. Any two of these operators can define unambiguously the hexagonal code, which is the intersection of their manifold with eigenvalue +1. In practice, our stabilization protocol employs symmetrically measurements of the three stabilizers in order for the peaks of the stabilized grid states to verify rotational symmetry. This is needed in order for the decay of the logical qubit Bloch vector to be isotropic (see Sec. S5.5).

The logical Pauli operators are $\mathbf{X} = \mathbf{D}(a/2)$, $\mathbf{Y} = \mathbf{D}(b/2)$ and $\mathbf{Z} = \mathbf{D}(c/2)$. The Wigner

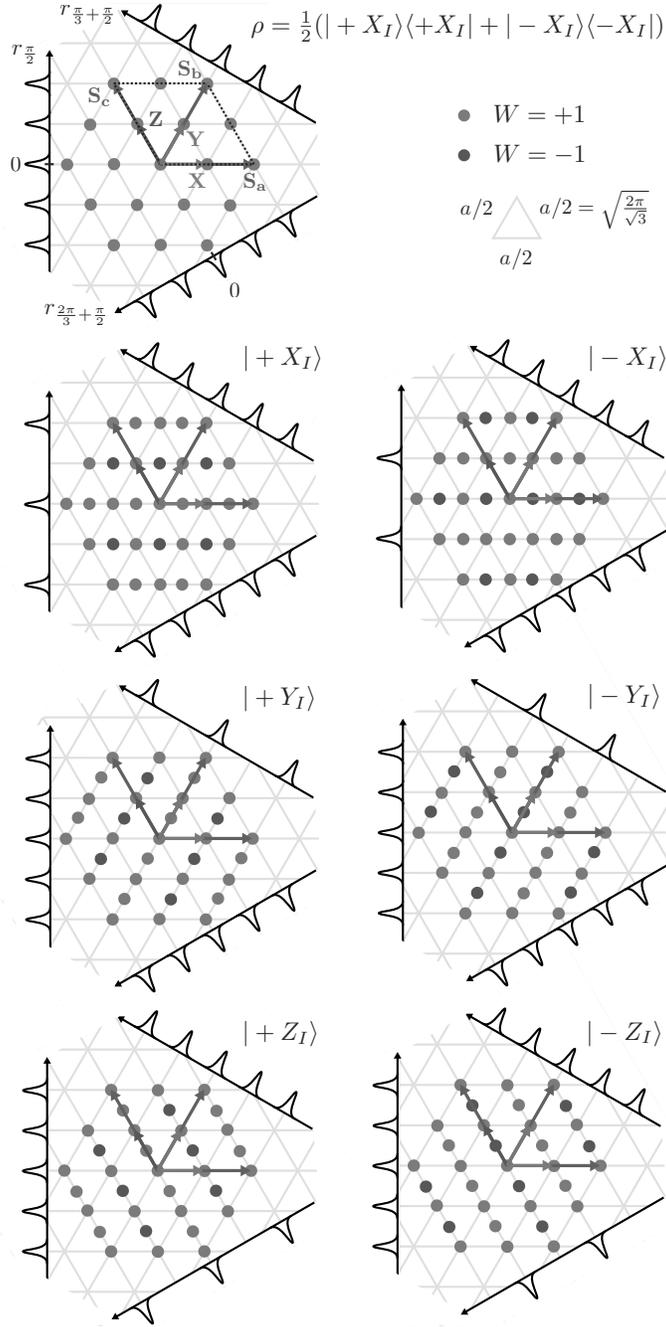


Figure S7: **Logical states in the ideal hexagonal code.** Wigner quasi-probability distributions and their marginals along axes orthogonal to a , b , and c are schematically represented for the maximally mixed logical state—revealing the hexagonal lattice structure, highlighted with gray lines—and the 6 Clifford states. We give quasi-probability peaks (red and blue dots) as well as peaks of the marginals (black lines) a finite extension for better visualization. The distribution extends infinitely in phase-space and is obtained by tessellating the pattern of the unit cell delimited by a black dashed line (area 4π).

function of the 6 Clifford states are represented in Fig. S7. From similar arguments given for the square code, we find that these functions are all a -periodic along r_0 , $r_{\pi/3}$ and $r_{2\pi/3}$, with an extra $a/2$ -periodicity along one of these axes for each Clifford state. Moreover, on the reciprocal lattice, the marginals P_θ with $\theta = \pi/2$, $\theta = \pi/3 + \pi/2$ and $\theta = 2\pi/3 + \pi/2$ of all states have non-zero values only at $r_\theta = 2\pi/a$. For the states $|\pm X_I\rangle$, $|\pm Y_I\rangle$ and $|\pm Z_I\rangle$, every other peak vanishes respectively for $\theta = \pi/2, \pi/3 + \pi/2, 2\pi/3 + \pi/2$.

S4 Finitely squeezed GKP code

In this section, we study the properties of the stabilized code in Wigner space. In order to avoid aberrations linked to maximum-likelihood reconstruction of the grid states Wigner functions from the measured characteristic functions, we extrapolate the code properties from master-equation simulations reproducing the data presented in the main paper with independently measured parameters. Thus, the figures of merit we find correspond to a realistic code whose width is optimized for the measured parameters of our experiment, but will not reflect the unknown experimental imperfections that are not captured by simulations.

S4.1 Optimal envelope size

In this section, we estimate the optimal envelope size for the GKP code in presence of storage oscillator photon loss only. Following Ref. (32), the dynamics of a harmonic oscillator entailed by photon damping at rate κ_s is modeled by a Fokker-Planck equation on its Wigner quasi-probability distribution. With the conventions used in the main text,

$$\frac{\partial W}{\partial t} = \frac{\kappa_s}{2} \left(\frac{\partial(qW)}{\partial q} + \frac{\partial(pW)}{\partial p} + \frac{1}{2} \frac{\partial^2 W}{\partial q^2} + \frac{1}{2} \frac{\partial^2 W}{\partial p^2} \right). \quad (\text{S5})$$

One can integrate this equation over, say, p to get an equation on the probability distribution $P(q)$

$$\frac{\partial P}{\partial t} = \frac{\kappa_s}{2} \left(\frac{\partial(qP)}{\partial q} + \frac{1}{2} \frac{\partial^2 P}{\partial q^2} \right). \quad (\text{S6})$$

We then have

$$\begin{aligned}
\frac{d\langle q \rangle}{dt} &= \int_q q \frac{\partial P}{\partial t} dq \\
&= \frac{\kappa_s}{2} \int_q q \frac{\partial(qP)}{\partial q} + \frac{1}{2} q \frac{\partial^2 P}{\partial q^2} dq \\
&= \frac{\kappa_s}{2} \left(- \int_q q P(q) dq + 0 \right) \\
&= -\frac{\kappa_s}{2} \langle q \rangle,
\end{aligned} \tag{S7}$$

and

$$\begin{aligned}
\frac{d\langle q^2 \rangle}{dt} &= \int_q q^2 \frac{\partial P}{\partial t} dq \\
&= \frac{\kappa_s}{2} \int_q q^2 \frac{\partial(qP)}{\partial q} + \frac{1}{2} q^2 \frac{\partial^2 P}{\partial q^2} dq \\
&= \frac{\kappa_s}{2} \int_q -2q^2 P(q) + P(q) dq \\
&= \kappa_s \left(-\langle q^2 \rangle + \frac{1}{2} \right).
\end{aligned} \tag{S8}$$

Thus, the first two terms in Eq. (S5) are drifts at speed $-\kappa_s q/2$ (deterministic contraction of the probability distribution with rate κ_s), while the last two correspond to diffusion with a constant $\kappa_s/2$ of the probability distribution. In steady-state, they compensate each other and the oscillator is in the vacuum state characterized by $\langle q \rangle = \langle p \rangle = 0$ and $\langle q^2 \rangle = \langle p^2 \rangle = \frac{1}{2}$.

Both drift and diffusion terms contribute to distorting the grid states and can lead to logical flips after some finite time. Indeed, the evolution of the quasi-probability distribution is continuous and the probability of a logical flip is the fraction of the distribution that has traveled by more than $a/4$ in phase-space. We estimate the optimal envelope width for the GKP code by requiring that the drift and the diffusion of the probability distribution result in the same average traveled distance during the typical time T of stabilization (defined more rigorously in the next paragraph), in the limit of short time $\kappa_s T \ll 1$. We consider a square grid state whose envelope width is Δ , and, for clarity, we consider the evolution of the q -probability distribution only. Over the time T , an infinitely squeezed state at position q drifts by a length

$d_{\text{drift}}(q) = |q|(1 - e^{-\kappa_s T/2})$ so that the average traveled distance for the whole distribution is

$$\begin{aligned} D_{\text{drift}} &= \int_q d_{\text{drift}}(q)P(q)dq \\ &= \Delta(1 - e^{-\kappa_s T/2}) \\ &\approx \Delta \frac{\kappa_s T}{2}. \end{aligned} \tag{S9}$$

Following Eq. (S8), photon dissipation also leads to a uniform diffusion of probability distribution in phase-space with a constant $\kappa_s/2$. Thus, an infinitely squeezed state at position q spreads over the time T to a peak with width $d_{\text{diff}}(q) = \sqrt{\frac{\kappa_s T}{2}}$. The average traveled distance for the whole distribution is

$$\begin{aligned} D_{\text{diff}} &= \int_q d_{\text{diff}}(q)P(q)dq \\ &= \sqrt{\frac{\kappa_s T}{2}}. \end{aligned} \tag{S10}$$

For the optimal envelope size, we get

$$\begin{aligned} D_{\text{drift}} = D_{\text{diff}} &= \sqrt{\frac{\kappa_s T}{2}} \\ \Delta &= \sqrt{\frac{2}{\kappa_s T}}. \end{aligned} \tag{S11}$$

These results can be understood with a simple model. We assume that after time-intervals of duration T , we perform instantaneous phase-estimation (15, 25, 26) of the stabilizers to project the oscillator state in a grid state with envelope Δ and peak width $\sigma = \frac{1}{2\Delta} = \frac{1}{2}\sqrt{\frac{\kappa_s T}{2}}$ (note that in general, we can have $\sigma\Delta > 1/2$ if the oscillator state has been randomly shifted out of the code manifold, which increases σ , while keeping Δ constant). During the next time interval, diffusion and drift of the quasi-probability inflate the peaks by $\sim \sqrt{\frac{\kappa_s T}{2}} = 2\sigma$ so that their width remains of order σ . Thus, the envelope size is optimal: if one were to change phase-estimation parameters to target grid states with thinner peaks (larger envelope), D_{drift} would increase and the peaks would re-inflate to a width larger than σ . On the other hand, if the target states have broader peaks (smaller envelope), orthogonal logical states overlap more than necessary, resulting in an increased logical flip rate. The shorter the stabilization time T , the thinner the peaks

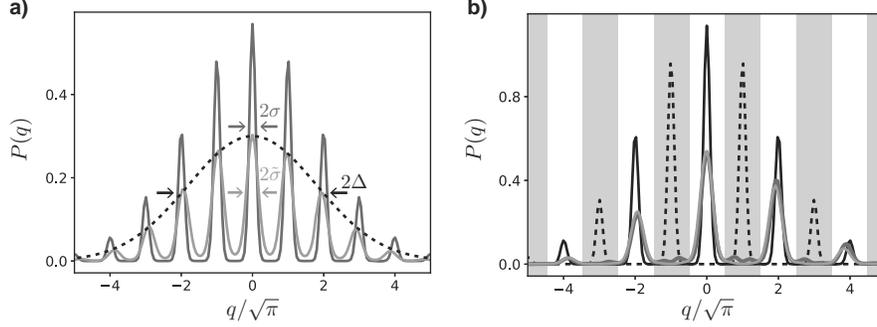


Figure S8: **Simulation of the experimentally stabilized code** **a)** The probability distribution of the steady-state of stabilization (orange line) has an envelope width $\Delta = 3.2$, chosen to maximize the logical qubit coherence time. The peaks are broadened to a width $\tilde{\sigma} = 0.29 > \sigma = 1/2\Delta$ by dissipation compared to the maximally mixed logical state strictly inside the code (gray line). **b)** Probability distribution of the state prepared when aiming for $|+X_L\rangle$. The blue line corresponds to a single $\text{Re}(\mathbf{X})$ measurement conditioning a feedback \mathbf{Z} -gate, and the orange line to the same protocol followed by a second $\text{Re}(\mathbf{X})$ measurement to herald a higher fidelity state. By integrating the distribution on $[-\sqrt{\pi}/2, \sqrt{\pi}/2] \bmod 2\sqrt{\pi}$ (gray stripes), we estimate that a perfect homodyne measurement along the q quadrature would assign the two prepared states to the target state $|+X_L\rangle$ with respective probability $F_2 = 0.9, 0.97$. However, the relative distance of these states to $|+X_L\rangle$ (plain black line) with respect to the orthogonal state $|-X_L\rangle$ (dashed black line) defines another preparation fidelity with higher value $F_3 = 0.976, 0.996$.

of the target states can be.

In the Markovian protocol used in the experiment, we can replace T by the characteristic convergence time $\sim 25 \mu\text{s}$ to reach steady-state when turning on the stabilization (see Fig. 3a). We then estimate an optimal envelope size $\Delta \sim 4$. This estimation only provides an order of magnitude as a rigorous treatment should include details of the stabilization scheme, higher moments of the probability distribution induced by drifts and other decoherence mechanisms beyond photon loss. We find quantitatively the optimal envelope width by maximizing the coherence time of the logical qubit in simulations, yielding $\Delta = 3.2$ (see Fig. S8a).

S4.2 Characteristics of the stabilized code

In this section, we summarize the properties of the experimentally stabilized GKP code, extracted from master-equation simulations (see Sec. S2.6). For simplicity sake, we consider the square code only during the remainder of this section. As noted in the main text, the properties of the stabilized hexagonal code are similar. In particular, the stabilized grid states are characterized by the same squeezing of the peaks for both encodings in steady state.

In Fig. S8a, we represent the q -probability distribution in steady-state of the stabilization (mixed logical state whose characteristic function is represented in Fig. 3b), right before a q -peak sharpening round. This is the time when the peaks of the q -probability distribution are the widest, and we would find the same distribution on p before a p -peak sharpening round. The grid state envelope has a width $\Delta = 3.2$, which is chosen to maximize the logical qubit coherence time. As detailed in Sec. S5.2, this width is set by the length of the conditional displacements used in envelope trimming rounds. The peaks of the grid state have a standard deviation $\tilde{\sigma} = 0.3 > 1/2\Delta$, which corresponds to a squeezing of $20 \text{ Log}_{10}\left(\frac{\delta q_{\text{vac}}}{\tilde{\sigma}}\right) = 7.4 \text{ dB}$ ($\delta q_{\text{vac}} = 1/\sqrt{2}$ is the standard deviation of the vacuum state probability distribution). Due to dissipation acting on the oscillator and other decoherence mechanisms, the peaks are thus broader than those of the actual code states, for which $\sigma = 1/2\Delta = 0.15$ (13.4 dB squeezing). This is consistent with the oscillator state purity which is smaller than that of the fully mixed logical qubit state $\text{Tr}(\rho_{ss}^2) = 0.17 < 0.5 = \text{Tr}(\rho_{\text{mix}}^2)$. As mentioned in the main text, the peaks of the distribution are thinner after a q -peak sharpening round, with a standard deviation corresponding to a squeezing of 9.5 dB.

We can now rigorously define the code states by

$$\begin{aligned}
\langle q| + X_L \rangle &\propto \sum_{n \in 2\mathbb{Z}} e^{-\frac{(q-n\sqrt{\pi})^2}{4\sigma^2} - \frac{n^2\pi}{4\Delta^2}} \\
\langle q| - X_L \rangle &\propto \sum_{n \in 2\mathbb{Z}+1} e^{-\frac{(q-n\sqrt{\pi})^2}{4\sigma^2} - \frac{n^2\pi}{4\Delta^2}} \\
\langle r| + Y_L \rangle &\propto \sum_{n \in 2\mathbb{Z}} e^{-\frac{(r-n\sqrt{\pi/2})^2}{4\sigma^2} - \frac{n^2\pi/2}{4\Delta^2}} \\
\langle r| - Y_L \rangle &\propto \sum_{n \in 2\mathbb{Z}+1} e^{-\frac{(r-n\sqrt{\pi/2})^2}{4\sigma^2} - \frac{n^2\pi/2}{4\Delta^2}} \\
\langle p| + Z_L \rangle &\propto \sum_{n \in 2\mathbb{Z}} e^{-\frac{(p-n\sqrt{\pi})^2}{4\sigma^2} - \frac{n^2\pi}{4\Delta^2}} \\
\langle p| - Z_L \rangle &\propto \sum_{n \in 2\mathbb{Z}+1} e^{-\frac{(p-n\sqrt{\pi})^2}{4\sigma^2} - \frac{n^2\pi}{4\Delta^2}}
\end{aligned} \tag{S12}$$

where $|q\rangle$, $|p\rangle$ and $|r\rangle$ respectively represent infinitely squeezed states along q , p and the axis r rotated from q at 45° . Note that in these expressions, the width of the wavefunction peaks is $\sqrt{2}\sigma$, larger than that of the probability distribution peaks. We estimate the overlap between two states representing orthogonal logical states to be $|\langle +X_L| - X_L \rangle|^2 = |\langle +Z_L| - Z_L \rangle|^2 \ll |\langle +Y_L| - Y_L \rangle|^2 = 4.10^{-7}$. This negligible overlap allows us to rigorously define a logical qubit in the finitely squeezed code.

S4.3 State preparation fidelity

We now use simulations described in Sec. S2.6 to estimate the preparation fidelity for the logical Clifford states (see Sec. 3). In Fig. S8b, we represent the simulated q -probability distribution of the state generated when aiming for $|+X_L\rangle$. The blue line represents the state ρ_1 obtained with a single-round measurement of $\text{Re}(\mathbf{X})$ followed by a feedback \mathbf{Z} -gate if the outcome is -1 . Again, note that \mathbf{X} corresponds to the Pauli operator of the ideal, infinitely squeezed GKP code, so that $\text{Re}(\mathbf{X})$ is strictly bi-valued only in that case. We reiterate that in the case of the finitely squeezed code, the distribution of the operator \mathbf{X} , whose complex eigenvalues lie on the unit circle, is still sharply peaked near $+1$ and -1 . Thus, extracting one bit of information is

sufficient to prepare *approximately* $|+X_L\rangle$ (plain black line on Fig. S8b). As described before, this measurement is performed with a σ_x readout of the transmon, following its preparation in $|+x\rangle$ and a conditional displacement $\text{CD}(a/2)$. Note that this conditional displacement is followed by a shift to recenter the grid at $q = 0 \bmod 2\pi/a$ (see Sec. S5.4). This shift, as well as the feedback \mathbf{Z} -gate, displaces the grid state envelope, which translates into an asymmetry of the q -probability distribution, or equivalently a non-zero value of the characteristic function imaginary part (not shown). Subsequent envelope-trimming steps corral the envelope back to the center of phase-space, restoring the symmetry of the grid.

We can give a first definition of the preparation fidelity as the expectation value of the real part of the ideal code Pauli operator, re-scaled to belong to $[0,1]$: $F_1 = (\langle \text{Re}(\mathbf{X}) \rangle + 1)/2 = 0.86$. This definition is relevant as it corresponds to the result of a subsequent logical qubit readout along \mathbf{X} as performed in the experiment (neglecting dissipation during the conditional displacement and transmon errors). However, it does not reflect fairly the preparation fidelity to the finitely squeezed target state $|+X_L\rangle$. Indeed, this fidelity would be smaller than one even when perfectly preparing the target state since the logical states that we consider here are not eigenstates of \mathbf{X} and thus $\langle +X_L | \text{Re}(\mathbf{X}) | +X_L \rangle < 1$. Similar arguments show that a protocol based on a single-round measurement of $\text{Re}(\mathbf{X})$ and feedback cannot perfectly prepare $|+X_L\rangle$.

A second definition of the preparation fidelity is given by integrating the q -probability distribution of the generated state over regions $[-\sqrt{\pi}/2, +\sqrt{\pi}/2] + 2n\sqrt{\pi}$ (gray stripes in Fig. S8b). We thus find the fidelity $F_2 = 0.90$. It corresponds to the probability that an ideal homodyne detection along the q quadrature, which collapses the grid state to an infinitely squeezed state, assigns the generated state to the target state $|+X_L\rangle$.

We can go one step further and consider the best possible guess in assigning the generated state to $|+X_L\rangle$ or $|-X_L\rangle$ from the full information of the simulated density matrix. This third

definition yields a higher fidelity to the target state $F_3 = \frac{\text{Tr}(\rho_1 | +X_L\rangle\langle +X_L|)}{\text{Tr}(\rho_1 | +X_L\rangle\langle +X_L|) + \text{Tr}(\rho_1 | -X_L\rangle\langle -X_L|)} = 0.976$. It is the value that one would get by performing a perfect projection of the state on the code manifold before measuring the bi-valued operator $(| +X_L\rangle\langle +X_L| + 1)/2$. Note that such an operation is for the moment hypothetical as it would require to measure the actual stabilizer of the finite width code, keeping in post-selection only the states belonging to the code manifold, before performing a single-shot measurement of the finite code Pauli operator $| +X_L\rangle\langle +X_L| - | -X_L\rangle\langle -X_L|$. This operator is rigorously defined since $| +X_L\rangle$ and $| -X_L\rangle$ are orthogonal up to a very good approximation (see Sec. S4.2), and it should also be possible to construct the stabilizers of the finite code, but no measurement protocol for these operators exists as of yet.

In the experiment, we boost the preparation fidelity by performing a second $\text{Re}(\mathbf{X})$ measurement and post-selecting the cases when the outcome is +1. We reproduce the generated state ρ_2 in simulations and, using the definitions given above, we find the preparation fidelities $F_1 = 0.90$, $F_2 = 0.97$ and $F_3 = 0.998$. The reason for $F_2 < F_3$ is that the peaks of the generated state when targetting $| +X_L\rangle$ (orange line on Fig. S8b) are broadened enough by dissipation for them to non-negligibly overlap with the regions colored in gray. Thus, a single-shot detection of q can assign the state to $| -X_L\rangle$ with non-negligible probability $1 - F_2$. However, the peak are not broad enough to significantly overlap with the actual pure logical state $| -X_L\rangle$ of the code (dashed black line on Fig. S8b), so that the thought-experiment described above, which includes post-selection, would assign the state to $| +X_L\rangle$ with higher probability F_3 . Interestingly, one could similarly obtain a higher preparation fidelity $F_{2'} \simeq F_3$ using an homodyne detection of q at the expense of post-selecting out the most ambiguous results when $q \bmod \sqrt{\pi} \simeq \sqrt{\pi}/2$ (obviously, this strategy cannot improve the *readout fidelity* of the logical qubit for this q -detection based scheme). Although $1 - F_3 = 0.2\%$, this metric is irrelevant for assessing quantum computational resources. However, it still gives us the indication that the state preparation fidelity is limited by the broadening of the grid states peaks only. The fact that

F_3 is close to unity indicates that preparation cannot further improved by repeating again the $\text{Re}(\mathbf{X})$ measurement.

Finally, it is important to note that the decay of the components of the logical qubit Bloch vector, as presented in Figs. 4-5, is independent of the used definition.

S5 Details of the stabilization protocol

S5.1 The conditional displacement gate

In this section, we compute explicitly the evolution induced by the Hamiltonian $\tilde{\mathbf{H}}$ (see Eq. (2)), which is approximated by a conditional displacement in Sec. 1. We note T_{int} the duration of interaction (pulse sequence preceding the readout in Fig. 2c), $\varepsilon = \frac{\chi T_{\text{int}}}{2N}$ and $t_k = k\varepsilon$. In the limit $N \rightarrow \infty$, we can write the evolution operator as

$$\mathbf{U} = \prod_{k=1}^N e^{i\varepsilon(\mathbf{H}_1(t_k) + \mathbf{H}_2(t_k) + \mathbf{H}_3(t_k))}, \quad (\text{S13})$$

where

$$\begin{aligned} \mathbf{H}_1(t) &= \mathbf{a}^\dagger \mathbf{a} s(t) \sigma_z \\ \mathbf{H}_2(t) &= (\alpha(t) \mathbf{a}^\dagger + \alpha(t)^* \mathbf{a}) s(t) \sigma_z \\ \mathbf{H}_3(t) &= |\alpha(t)|^2 s(t) \sigma_z \\ s(t) &= 1 - 2\Theta(t - T_{\text{int}}/2) \end{aligned} \quad (\text{S14})$$

In these expressions, $s(t) = \pm 1$ accounts for the transmon echo at $T_{\text{int}}/2$ (see Fig. 2b), and we use the ordering $\prod_{k=1}^N \mathbf{U}_k = \mathbf{U}_N \mathbf{U}_{N-1} \dots \mathbf{U}_1$ for the product so that the evolution operators are applied in chronological order. The terms $\mathbf{H}_3(t_k)$ commute with $\mathbf{H}_1(t_l)$, $\mathbf{H}_2(t_l)$ and $\mathbf{H}_3(t_l)$ at all times t_l , and can thus be factored out of the exponential and grouped in a separated product. Moreover, $\prod_{k=1}^N e^{i\varepsilon \mathbf{H}_3(t_k)} = \mathbf{I}$ so that the corresponding evolution can be dropped out. We now use the Baker-Campbell-Hausdorff formula to expand the exponential terms in $\mathbf{H}_1 + \mathbf{H}_2$ at each

time t_k

$$\mathbf{U} = \prod_{k=1}^N e^{i\varepsilon \mathbf{H}_1(t_k)} e^{i\varepsilon \mathbf{H}_2(t_k)} \mathbf{U}_{nc}(t_k), \quad (\text{S15})$$

where \mathbf{U}_{nc} captures an infinite product of nested commutators exponentials. We verify that $\log(\mathbf{U}_{nc}(t_k)) = O(\varepsilon^2)$ so that these evolutions can be neglected when integrating over $[0, T_{\text{int}}]$.

We use again the Baker-Campbell-Hausdorff formula to rearrange the remaining terms in Eq. (S15):

$$\mathbf{U} = \left(\prod_{k=1}^N \prod_{l=1}^{k-1} e^{-\varepsilon^2 \mathbf{G}(t_l)} \right) \left(\prod_{k=1}^N e^{i\varepsilon \mathbf{H}_2(t_k)} \right) \left(\prod_{k=1}^N e^{i\varepsilon \mathbf{H}_1(t_k)} \right). \quad (\text{S16})$$

Here, we have defined $\mathbf{G}(t) = [\mathbf{H}_1(t), \mathbf{H}_2(t)] = \alpha(t)\mathbf{a}^\dagger - \alpha^*(t)\mathbf{a}$, and commuted the terms $e^{-\varepsilon^2 \mathbf{G}(t_l)}$ through the others by neglecting $N^2/2$ evolution operators $\mathbf{V}_{nc}(t_k, t_l)$ with $\log(\mathbf{V}_{nc}(t_k)) = O(\varepsilon^3)$.

Now using that $[\mathbf{H}_1(t_k), \mathbf{H}_1(t_l)] = [\mathbf{H}_2(t_k), \mathbf{H}_2(t_l)] = [\mathbf{G}(t_k), \mathbf{G}(t_l)] = 0$ for all k, l (α keeps the same phase modulo π over the whole evolution) and going to the continuous limit we get

$$\mathbf{U} = e^{-\frac{\chi^2}{4} \int_{t=0}^{T_{\text{int}}} \int_{t'=0}^t \mathbf{G}(t') dt' dt} e^{i\frac{\chi}{2} \int_{t=0}^{T_{\text{int}}} \mathbf{H}_2(t) dt} e^{i\frac{\chi}{2} \int_{t=0}^{T_{\text{int}}} \mathbf{H}_1(t) dt}. \quad (\text{S17})$$

The last term in this product cancels due to the transmon echo at $T_{\text{int}}/2$. The second term gives the conditional displacement $\mathbf{CD}(\beta)$ with $\beta = i\sqrt{2}\chi \left(\int_t^{T_{\text{int}}/2} \alpha(t) dt - \int_{T_{\text{int}}/2}^{T_{\text{int}}} \alpha(t) dt \right)$ that we consider in Sec. 1. The first term gives a short unconditional displacement of the oscillator orthogonal to the desired conditional displacement. We finally get

$$\mathbf{U} = \mathbf{D}(\gamma) \mathbf{CD}(\beta), \quad (\text{S18})$$

with $\gamma = -\frac{\chi^2}{2\sqrt{2}} \int_{t=0}^{T_{\text{int}}} \int_{t'=0}^t \alpha(t') dt' dt$. Note that in the limit of a fast conditional displacement for which $\int |\alpha| dt \rightarrow \infty$ and $T_{\text{int}} \rightarrow 0$, we have $|\gamma/\beta| \rightarrow 0$.

For the values of α and the interaction time we use for the square grid stabilization, we find $|\gamma| = 0.04 \sim |\beta|/60$. This small spurious shift of the oscillator state is compensated for when applying the feedback displacement following the interaction (see Fig. 2c), which is also

in the orthogonal direction to the conditional displacement. The feedback shifts after peak-sharpening steps are thus asymmetric and read $\delta_q = +0.04 \pm 0.2$ when sharpening the peaks along q , and $\delta_p = -0.04 \mp 0.2$ when sharpening the peaks along p . In practice, we vary the length of these corrective shifts around the predicted value and find $\delta_q = +0.06 \pm 0.2$ and $\delta_p = -0.06 \mp 0.2$ to empirically maximize the logical qubit coherence time (see Sec. S5.3). This small discrepancy can be attributed to experimental imperfections shifting the grid states during the other stabilization rounds: for comparison, a relative variation by $\sim 10^{-3}$ of the voltage of the DAC used to generate the storage mode displacements pulses (see Fig. S1) between the first and second half of the conditional displacements is sufficient to create such systematic shifts.

S5.2 Markovian feedback stabilization in the square code

We now write explicitly the Kraus operators acting on the oscillator state when measuring the transmon on a round of stabilization. Each round starts with a preparation of the transmon in $|+x\rangle$ (see Fig. 2b). We re-write the following conditional displacement as $\text{CD}(\beta) = e^{i\beta_{\perp} \frac{\sigma_z}{2}}$, where we have defined $\beta_{\perp} = -\text{Re}(\beta)\mathbf{p} + \text{Im}(\beta)\mathbf{q}$. Finally, the transmon is measured along σ_y . If we neglect decoherence during the sequence and the short unconditional displacement described in the previous section, the Kraus operators acting on the storage mode associated with a transmon detection in $|\pm y\rangle$ is

$$\begin{aligned} \mathbf{M}_{\pm} &= \langle \pm y | e^{i\beta_{\perp} \frac{\sigma_z}{2}} | +x \rangle \\ &= \langle +x | e^{i(\beta_{\perp} \pm \frac{\pi}{2}) \frac{\sigma_z}{2}} | +x \rangle \\ &= \cos\left(\left(\beta_{\perp} \pm \frac{\pi}{2}\right)/2\right). \end{aligned} \tag{S19}$$

In Fig. S9, we represent the backaction of the transmon readout, including effect of decoherence during the transmon-oscillator interaction, found using the master-equation simulations described in Sec. S2. After a large number of stabilization rounds (steady-state of Fig. 3), we consider a q -peak sharpening round of the square code for which $\beta_{\perp} = a\mathbf{q}$ (top panel). The black line represents the q -probability distribution $P_0(q) = \int_p W_0(q, p) dp$ for the oscillator state ρ_0 right before performing the sharpening round. The plain red and blue curves represent the

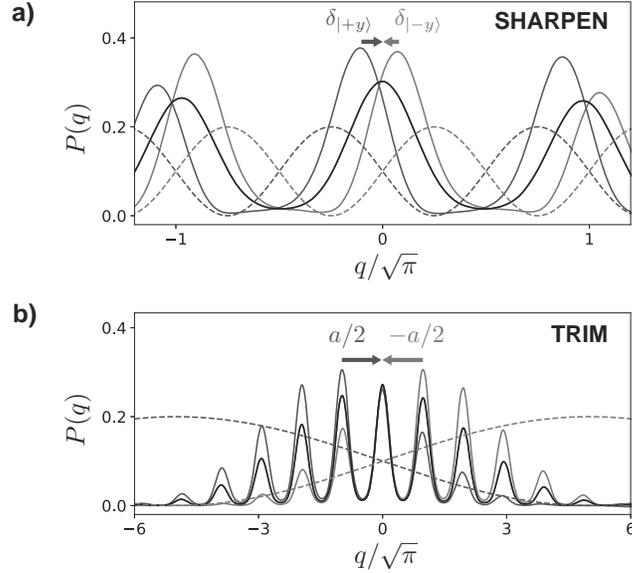


Figure S9: **Simulated backaction of the transmon readout on the oscillator probability distribution.** **a)** During a q -peak sharpening round occurring in steady-state (logical qubit in a fully mixed state), the backaction of the measurement of the transmon in $|\pm y\rangle$ multiplies the initial q -probability distribution (black line) by $\cos^2((qa \pm \frac{\pi}{2})/2)$ (blue and red dashed lines, scaled down for convenient representation). The peaks thus get sharper and offset (blue and red plain lines). When measuring the transmon in $|+y\rangle$ or $|-y\rangle$, this offset is asymmetric with respect to the origin due to a displacement term appearing in the oscillator state evolution during the sharpening round (see Sec. S5.1). Thus, the feedback shifts (arrows) re-centering the grid back towards $q = 0 \bmod 2\pi/a$ are asymmetric and read $\delta_{|\pm y\rangle} = \pm 0.2 + 0.04$. **b)** During a q -envelope trimming step, the initial distribution is multiplied by $\cos^2(q(\epsilon \pm \frac{\pi}{2})/2)$ with $\epsilon \approx a/20$ (same color code). The probability distribution is partly collapsed towards positive or negative values and a feedback displacement by $a/2$ shifts it back towards the origin without changing the code stabilizers value.

probability distribution $P_{\pm}(q)$ of the state ρ_{\pm} after performing the conditional displacement and measuring the transmon in $|\pm y\rangle$ (neglecting decoherence during the interaction and the small unconditional displacement described in the previous section, $\rho_{\pm} = \frac{M_{\pm}\rho_0M_{\pm}^{\dagger}}{\text{Tr}(M_{\pm}\rho_0M_{\pm}^{\dagger})}$). The dashed lines represent $\cos^2\left((qa \pm \frac{\pi}{2})/2\right)$ (scaled down for convenient representation). Qualitatively, we verify that $P_{\pm}(q) \propto P_0(q) \cos^2\left((qa \pm \frac{\pi}{2})/2\right)$. The measurement backaction thus partly collapses the peaks of the q -probability distribution, which get sharpened. Their center of mass are offset, which is corrected for by a small feedback displacement (red and blue arrows). The peaks also become skewed, but quickly retrieve a Gaussian shape under the effect of dissipation.

One also needs to consider the measurement backaction on the p -probability distribution (not represented). The Kraus operators of Eq. (S19) can be re-written as

$$\mathbf{M}_{\pm} = \frac{1}{2} \left(e^{\pm i\frac{\pi}{4}} \mathbf{D}(\beta/2) + e^{\mp i\frac{\pi}{4}} \mathbf{D}(-\beta/2) \right), \quad (\text{S20})$$

with $\beta = b = ia$ for a q -peak sharpening round. Along p , the probability distribution expands with the generation of a new outward peak. It also gets shifted by $\frac{a}{2}$ modulo a , which deterministically flips the logical qubit (**Z**-gate for the q -peak sharpening rounds, **X**-gate for the p -peak sharpening rounds). Finally, interference between the two displaced versions of the state with a relative $\frac{\pi}{2}$ -phase distorts the p -probability distribution envelope, which can be seen as a consequence of the skewness acquired by the q -peaks by virtue of the Fourier transform properties.

To mitigate this undesired expansion of the envelope, we alternate stabilization rounds dedicated to sharpening the peaks with rounds dedicated to trimming the envelope. In this case, we use a shorter conditional displacement $\mathbf{CD}(\epsilon)$, with $|\epsilon| \approx a/20$. In Fig. S9b, we represent the backaction of the transmon σ_y readout at the end of a q -envelope trimming step for which $\beta_{\perp} = \epsilon \mathbf{q}$. Similarly to the peak sharpening step, this backaction can be understood as a multiplication of $P(q)$ by $\cos^2\left((q\epsilon \pm \frac{\pi}{2})/2\right)$, which partly collapses the distribution towards positive or negative q -values. A following feedback displacement (blue and red arrows) shifts the whole distribution back towards the origin. These displacements by $a/2$ are large enough to com-

compensate for the envelope expansion induced by peak-sharpening rounds. They commute with the code stabilizers and thus do not modify their values. The logical qubit is deterministically flipped (the feedback displacement performs a X -gate for the q -envelope trimming rounds, and a Z -gate for the p -envelope trimming rounds).

The value of ϵ sets the “strength” of the measurement. When $\epsilon \rightarrow 0$, the multiplying cosine has a period much larger than the represented q -probability distribution. The backaction would then only marginally modify the state, and the envelope would keep expanding during peak-sharpening rounds until it reaches a steady-state with a much larger envelope. In the opposite limit, a larger ϵ value would result in a stronger collapse of the probability distribution and a smaller envelope in steady-state.

S5.3 Optimization of the feedback displacements

The optimal length of the feedback displacements applied at the end of each stabilization round is first estimated in simulations, and then finely adjusted by maximizing the real part of the stabilizers expectation value in steady-state.

The general procedure is the following. Starting from an initial guess for all the feedback shifts, we apply stabilization for a large number of rounds in order to reach steady-state. For the last stabilization round only, we vary independently the length of the shift applied when the transmon is detected in $|+y\rangle$ or $|-y\rangle$. Conditioned on this last measurement outcome, we then measure the expectation value of the stabilizers (this measurement is performed by resetting the transmon and averaging the result of a subsequent measurement of S_a or S_b as described in Sec. 1). The conditional expectation values $\langle S_a \rangle_{|\pm y\rangle}$ when tuning the feedback shift at the end of a p -peak sharpening round are represented on Fig. S10a-b. Varying the length δ of a shift displaces the generated grid state along p , which rotates the phase of $\langle S_a \rangle_{|\pm y\rangle}$ by an angle $a\delta$.

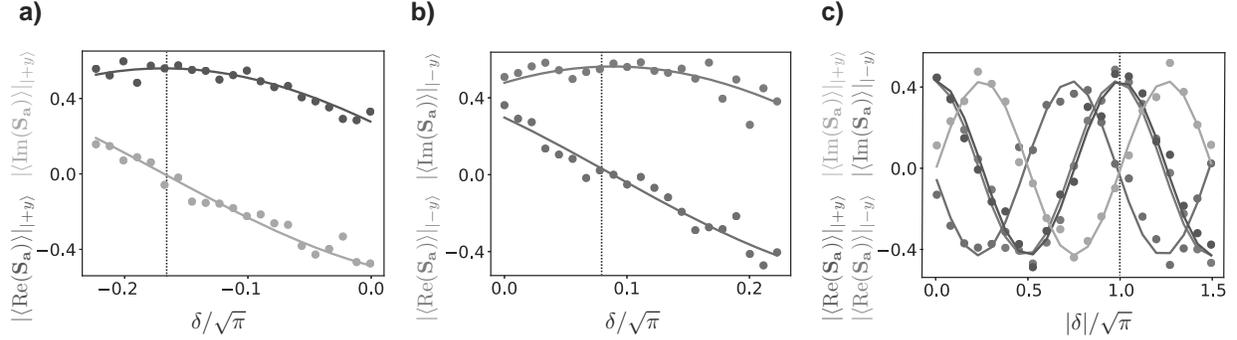


Figure S10: **Tuning of the feedback displacements.** **a)** After reaching steady-state of the stabilization, we perform a last p -peak sharpening round for which we vary the length δ of the feedback shift when the transmon is measured in $|+y\rangle$, and record the stabilizer expectation value $\langle \mathbf{S}_a \rangle_{|+y\rangle}$ conditioned on this same outcome (dots). The phase of $\langle \mathbf{S}_a \rangle_{|+y\rangle}$ oscillates with a period $2\pi/a$ (lines represent a *sine* and *cosine* function with this period), and the optimal feedback shift (vertical dashed line) cancels its imaginary part. The grid state is then centered in $p \bmod 2\pi/a = 0$. **b)** Idem when measuring the transmon in $|-y\rangle$. **c)** Same measurement after a p -envelope trimming round. The optimal feedback shifts length is $a/2 = 2\pi/a$.

We fit the recorded real and imaginary parts (circles) with a *cosine* and a *sine* function (lines) with period $2\pi/a$. The phase-offset of these oscillations directly indicates the optimal feedback shift (vertical dashed line) following a transmon measurement in $|+y\rangle$ or $|-y\rangle$: after applying these shifts, $\langle \mathbf{S}_a \rangle$ becomes real and the grid state is centered in $p \bmod 2\pi/a = 0$. Note that the shifts corresponding to the $|\pm y\rangle$ outcomes are asymmetric, which is expected (see Sec. S5.1).

We perform the same tuning for the feedback shifts following a p -envelope trimming round (see Fig. S10c). Here, as expected, the optimal shifts are symmetric and their length is close to $a/2$ (enforcing that they are strictly $a/2$ can be seen as a more precise calibration of the displacements length compared to the method described in Sec. S2.2). We also tune the shifts following q -peak sharpening and q -envelope trimming rounds (nullifying $\langle \mathbf{S}_b \rangle_{|\pm y\rangle}$, not shown). Since the steady-state grid used as an input to this optimization uses an imprecise initial guess for the feedback shifts and can thus be offset, we then iterate the whole protocol. The shifts

are adjusted one final time to maximize the Pauli operators lifetime, and we obtain the optimal shift value $\delta_q = +0.06 \pm 0.2$ after a q -peak sharpening round and $\delta_p = -0.06 \mp 0.2$ after a p -peak sharpening round, in good agreement with our model for conditional displacements (see Sec. S5.1). Note that for simplicity, we do not mention the shifts asymmetry in the main text.

S5.4 Measurement of the logical Pauli operators in the square code

Measurement of $\text{Re}(\mathbf{X})$ (respectively $\text{Re}(\mathbf{Y})$ and $\text{Re}(\mathbf{Z})$) is performed by replacing a q -peak sharpening step with a conditional displacement $\mathbf{CD}(\beta)$ with $\beta = a/2$ (respectively $(a + b)/2, b/2$) followed by a σ_x readout of the transmon. The Kraus operators corresponding to the two measurement outcomes are

$$\begin{aligned} N_+ &= \cos(\beta_\perp/2) = \frac{1}{2} \left(\mathbf{D}(\beta/2) + \mathbf{D}(-\beta/2) \right) \\ N_- &= \sin(\beta_\perp/2) = \frac{1}{2i} \left(\mathbf{D}(\beta/2) - \mathbf{D}(-\beta/2) \right) \end{aligned} \quad (\text{S21})$$

where $\beta_\perp = -\frac{a}{2}\mathbf{p}$ (respectively $\beta_\perp = \frac{a}{2}(\mathbf{q} - \mathbf{p})$, $\beta_\perp = \frac{a}{2}\mathbf{q}$). The measurement backaction thus multiplies the β_\perp -probability distribution by $\frac{1 \pm \cos \beta_\perp}{2}$, which damps one peak of the distribution out of two. On the conjugate variable, it displaces the state by $\pm\beta/2$, which reverses the sign of one or both of the stabilizers (shifts the grid state by half the lattice period). We flip back the stabilizer sign by applying an unconditional displacement $\mathbf{D}(-\beta/2)$ at the end of sequence.

S5.5 Stabilization and measurement of the Pauli operators in the hexagonal code

We recall that the hexagonal code words are approximate eigenstates with eigenvalue 1 of the stabilizers $\mathbf{S}_a = \mathbf{D}(a = \sqrt{\frac{8\pi}{\sqrt{3}}})$, $\mathbf{S}_b = \mathbf{D}(b = ae^{i\frac{\pi}{3}})$ and $\mathbf{S}_c = \mathbf{D}(c = ae^{2i\frac{\pi}{3}})$. These stabilizers are redundant for the infinitely squeezed code as any two of them can define it unambiguously, but as explained below, our stabilization protocol based on measurement of two stabilizers only would result in grid states with peaks without rotational symmetry. Then, the lifetime of the

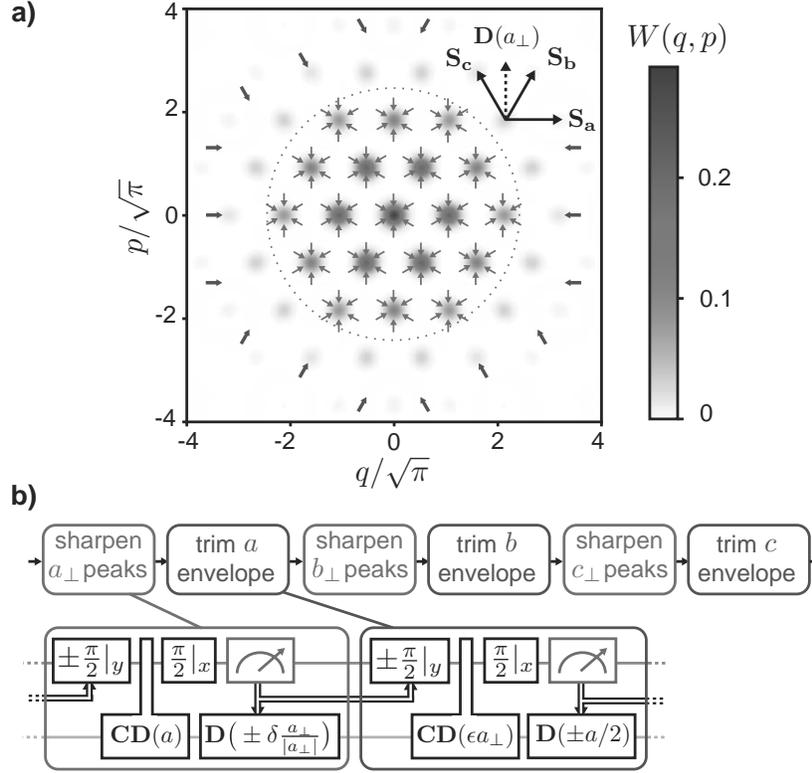


Figure S11: **Hexagonal code stabilization.** **a)** Simulated Wigner function of the logical mixed state for an hexagonal code with similar envelope width as in the experiment. The Markovian feedback stabilization generates position and momentum dependent shifts from 6 directions: displacements along a_{\perp} , b_{\perp} and c_{\perp} prevent the grid state peaks from spreading (blue arrows), while displacements along a , b and c prevent the envelope from expanding (purple arrows). **b)** Repeated stabilization sequence. Peak-sharpening steps consist of conditional displacements with same direction and length as the displacements stabilizing the code, followed by a transmon measurement triggering short feedback displacements on the orthogonal direction. An envelope-trimming step consists of a short conditional displacement orthogonal to the axes of the stabilizers, followed by a transmon readout triggering a long feedback displacement (half a stabilizer length).

three Pauli operators $\mathbf{X} = \mathbf{D}(a/2)$, $\mathbf{Y} = \mathbf{D}(b/2)$ and $\mathbf{Z} = \mathbf{D}(c/2)$ would not be equal.

The Markovian feedback protocol is directly adapted from the square code case, and is schematized in Fig. S11. The peaks of the grid state quasi-probability distribution are sharpened from the three directions $a_{\perp} = ae^{i\frac{\pi}{2}}$, $b_{\perp}e^{i\frac{\pi}{2}}$ and $c_{\perp}e^{i\frac{\pi}{2}}$ orthogonal to the stabilizers displacement. Indeed, as detailed in Sec. S5.2, the peak-sharpening step corresponding to the measurement of \mathbf{S}_{β} ($\beta = a, b, c$) is based on the conditional displacement $\mathbf{CD}(\beta) = e^{i\beta_{\perp}\frac{\sigma_z}{2}}$, with $\beta_{\perp} = -\text{Re}(\beta)\mathbf{p} + \text{Im}(\beta)\mathbf{q}$. The effect of this entangling gate is to map the oscillator state probability distribution $P(\beta_{\perp}) = \int_{\beta} W(\beta, \beta_{\perp})d\beta$ onto the statistics of the transmon Bloch vector longitudinal angle. The subsequent transmon σ_y measurement backaction multiplies $P(\beta_{\perp})$ by an oscillating function which sharpens its peaks by partly collapsing them. A feedback shift along β_{\perp} with length $\pm\delta = \pm 0.2$ recenters the peaks at $\beta_{\perp} = 0 \bmod \frac{2\pi}{k}$. Overall, the repeated feedback shifts generate an effective dissipation preventing the peaks from spreading (blue arrows in Fig. S11a). Note that the conditional displacement $\mathbf{CD}(a)$, $\mathbf{CD}(b)$, $\mathbf{CD}(c)$ applied during each stabilization round flips the logical qubit in a deterministic way (\mathbf{X} , \mathbf{Y} , \mathbf{Z} -gate).

As represented on Fig. S11, the repetition of peak sharpening steps generates position and momentum dependent kicks in phase-space along a_{\perp} , b_{\perp} and c_{\perp} . It is here clear that the same stabilization protocol based on two stabilizers measurement only would result in ellipse-shaped peaks. For instance, not sharpening the peaks along $a_{\perp} = p$ would result in peaks more elongated along this direction than along the orthogonal direction q .

To trim the envelop of the grid state, In practice, we perform short conditional displacements along a_{\perp} , b_{\perp} , c_{\perp} followed by transmon readout along σ_y and feedback displacements respectively by $\pm a/2$, $\pm b/2$, $\pm c/2$. These trimming directions are chosen so that feedback displacements do not modify the stabilizers value, even if they flip the logical qubit in a deterministic way (applying a \mathbf{X} , \mathbf{Y} and \mathbf{Z} gate respectively). As in the square code case, the

feedback displacements could be made longer by an integer factor with no added complexity, but simulations indicate that these choices would result in a larger envelope in steady-state and a shorter logical qubit coherence time.

Finally, let us note that similarly to the initialization of the square code logical qubit described in Sec. S5.4, measurement of $\text{Re}(\mathbf{X})$, $\text{Re}(\mathbf{Y})$ or $\text{Re}(\mathbf{Z})$ is done by modifying a peak-sharpening round of the corresponding stabilizer \mathbf{S}_a , \mathbf{S}_b or \mathbf{S}_c : we set the conditional displacement to be twice shorter as $\text{CD}(a/2)$, $\text{CD}(b/2)$ and $\text{CD}(c/2)$, and the transmon is readout along σ_x . An additional unconditional displacement $\mathbf{D}(-a/2)$, $\mathbf{D}(-b/2)$, $\mathbf{D}(-c/2)$ is applied to recenter the grid state, which is otherwise shifted by half a stabilizer period by the conditional displacement.

S6 Transmon errors and feedback performances

In this section we estimate the convergence rate towards the code manifold under stabilization and discuss the impact of transmon errors on our stabilization scheme. For simplicity sake, we reason in the framework of the square code, but the qualitative results which are given also apply for the hexagonal code, unless otherwise noted.

S6.1 Readout errors and phase-flips of the transmon

The interaction Hamiltonian used to engineer the conditional displacements acts on the transmon via the σ_z operator only (see Eq. (2)). Phase-flips of the transmon (σ_z -gates applied at random times) thus commute with the interaction Hamiltonian: they have the same effect on the system that they occur before, during, or after the interaction. In our Markovian feedback protocol, transmon phase-flips are thus equivalent to transmon readout errors and simply lead to a feedback displacement applied in the wrong direction. We prove in this section that the convergence rate toward the stabilized manifold decreases only linearly with the probability of error: if this rate remains much larger than the dissipation rate κ_s , the stabilization performance

is only marginally affected.

To prove this, let us suppose that there exists in the oscillator a highly squeezed GKP state (peak width $\sigma \ll a$). We define $x = \left((q + \pi/a) \bmod 2\pi/a \right) - \pi/a$. The distribution $P(x)$ is a centered Gaussian with width σ . We note that the p -peak sharpening rounds and the envelope trimming rounds do not modify $P(x)$ (in the limit of large envelope). Moreover, during the q -sharpening rounds, the measurement backaction (multiplication by $\cos(\frac{a}{2}x \pm \frac{\pi}{4})$ according to Eq. (S19)) does not modify x in the limit of large squeezing. x can thus be understood as encoding the position of a classical particle performing a 1D random walk. In absence of transmon phase-flips, its position is shifted each q -sharpening round by $\pm\delta$, with respective probability

$$\begin{aligned} P_{\pm}^0(x) &= \cos^2\left(\frac{a}{2}x \pm \frac{\pi}{4}\right) \\ &\approx \frac{1}{2}(1 \mp 2ax), \end{aligned} \tag{S22}$$

where we have used that $x \ll a$. Now considering phase-flips of the transmon or readout errors taking place with probability ϵ each round, we get the new probabilities for the left and right jumps of the particle

$$P_{\pm}(x) = \frac{1}{2}(1 \mp (1 - 2\epsilon ax)). \tag{S23}$$

The continuous version of this random walk is a process corresponding to a Fokker-Planck equation on the distribution $P(x)$

$$\frac{\partial P}{\partial t} = \Gamma \frac{\partial(xP)}{\partial x} + D \frac{\partial^2 P}{\partial x^2} \tag{S24}$$

with $\Gamma = \delta(1 - 2\epsilon)a/\tau$ and $D = \delta^2/\tau$ (for $x \ll a$). Here, $\tau = 4T_{\text{round}}$ is the time between two q -peak sharpening rounds. Similarly to the case of photon dissipation (see Eq. (S6)), the distribution $P(x)$ converges with rate Γ toward a Gaussian centered in 0 and with variance $\sigma_{\infty}^2 = \frac{D}{\Gamma} = \frac{\delta}{(1-2\epsilon)a}$. Thus, in the limit $\delta \rightarrow 0$, one can achieve infinite squeezing of the GKP state despite an arbitrary large phase-flip rate or an arbitrary low transmon readout fidelity (as

long as this readout does provide some information so that $\epsilon > 0.5$).

Let us comment on this result.

- This short feedback displacements strategy is qualitatively equivalent to performing phase-estimation (15, 25, 26) of the stabilizers based on a great number of redundant transmon measurements before applying a feedback displacement: the repeated transmon measurements mitigate any infidelity of the readout and allow one to achieve arbitrary precision on the phase-estimation.
- The convergence rate Γ toward the code manifold is proportional to the transmon readout contrast $1 - 2\epsilon$. It is thus simply linear in the information extraction rate from the system.
- $\Gamma \propto \delta$ so that by decreasing δ to reduce σ_∞ , one sacrifices on the stabilization rate. When considering photon loss at rate κ_s in Eq. (S24), one gets the steady-state peak width $\sigma \approx \sqrt{\frac{D+\kappa_s/2}{\Gamma}}$. Here, we have used an effective diffusion coefficient $\kappa_s/2$ for the photon dissipation, which assumes an optimal choice of envelope size (see Sec. S4.1). For a given stabilization time T_{round} , the optimal value of δ is thus the result of a trade-off: long displacements are desired to counteract dissipation-induced diffusion, but need to be shortened when the transmon readout is inaccurate.
- Following Eq. (S20), the envelope trimming rounds (respectively in q, p) can be seen as conditional displacements by ϵ , and also contribute to broadening the peaks (respectively in p, q).

In our experiment, the transmon pure dephasing rate is $\Gamma_\phi = (1/T_2 - 1/2T_1)/2 = (140 \mu\text{s})^{-1}$ and readout errors are negligible (readout fidelity above 99.5 %). As can be seen from Table. 2, the impact of transmon phase-flips on the error-correction performances is negligible: with the stabilization parameters used in the experiment, simulations indicate that they only account for ~ 1.5 % of the logical errors.

S6.2 Bit-flips of the transmon

Contrary to phase-flips, bit-flips of the transmon (random σ_x -gate) do not commute with the interaction Hamiltonian and can thus perturb the oscillator state. Let us for now set aside the details of our protocol for performing conditional displacements, and assume that we have a controllable interaction Hamiltonian $\mathbf{H} = g(t)\mathbf{r}\sigma_z$ (with $\mathbf{r} = \mathbf{q}$ or \mathbf{p}). We note T_{int} the duration of the transmon oscillator interaction ($T_{\text{int}} = 1.1 \mu\text{s}$ in the experiment, see Fig. 2b). For the peak sharpening rounds, we have $\int_{t=0}^{T_{\text{int}}} s(t)g(t)dt = a/2$, where $s(t) = \pm 1$ accounts for the transmon echo pulse at $T_{\text{int}}/2$. During these rounds, the oscillator state is conditionally displaced by $\pm a/2$. As mentioned in Sec. S5.2, the logical qubit is thus deterministically flipped (the conditional displacement by $\pm a/2$ performs an unconditional \mathbf{X} -gate when sharpening the peaks along p , and the conditional displacement by $\pm b = \pm ia$ a \mathbf{Z} -gate when sharpening along q). If a transmon bit-flip occurs during the interaction, $s(t)$ changes sign at a random time and part of the evolution is canceled. The oscillator state is then displaced out of the code by $\pm d$ with $d < a/2$. Subsequent stabilization rounds recenter the grid state in the code manifold, but if $d < a/4$, a logical flip has occurred (the expected \mathbf{X} or \mathbf{Z} -gate has not been applied). If on the other hand $d > a/4$ or if the transmon bit-flip happens after the interaction with the oscillator, no logical flip occurs (the following feedback displacement can still be in the wrong direction similarly to what happens after a transmon phase-flip).

Therefore, transmon bit-flips induce \mathbf{X} (respectively \mathbf{Z} , \mathbf{Y}) logical errors only if they take place during a p -peak sharpening round (respectively a q -peak sharpening round, a p -peak or q -peak sharpening round) at a time t with $\frac{T_{\text{int}}}{4} \lesssim t \lesssim \frac{3T_{\text{int}}}{4}$ (see Fig. 2b, the inequality would be exact if $|g(t)|$ were constant). Moreover, in the experiment, the transmon decays to a cold bath so that bit-flips originate from an amplitude damping channel at rate $1/T_1$. Regular echoes in the stabilization sequence (see Fig. 2b) tend to symmetrize this channel, but the average occupation of the excited level is only 0.5, so that bit-flips happen at rate $1/2T_1$. We can then estimate the rates of logical flips induced by transmon bit-flips as $\Gamma_X = \Gamma_Y/2 = \Gamma_Z \approx \frac{16}{T_1} \frac{T_{\text{int}}}{T_{\text{round}}}$, which

limit the lifetime of the three components of the logical Bloch vector to $T_X, T_Z \lesssim 8T_1 \frac{T_{\text{round}}}{T_{\text{int}}}$ and $T_Y \lesssim 4T_1 \frac{T_{\text{round}}}{T_{\text{int}}}$. With a similar reasoning, we can show that $T_X, T_X, T_Z \lesssim 6T_1 \frac{T_{\text{round}}}{T_{\text{int}}}$ for the hexagonal code.

This picture is only slightly modified when considering the real interaction Hamiltonian (see Eq. (2)). In that case, beyond the random displacements that we just described, transmon bit-flips also lead to spurious rotations of the oscillator state. Such rotations arise from the dispersive transmon-oscillator coupling, which are normally echoed out by regular π -rotations of the transmon applied every $T_e \sim T_{\text{int}}/2$. A transmon bit-flip acts as a supplementary echo which breaks the regularity of the echo sequence and can result in a rotation by an angle $\phi \leq \chi T_e \approx 10^\circ$ in our experiment. These rotations increase slightly the logical error rate, but could be mitigated by performing more frequent echoes of the transmon during the conditional displacements (reversing accordingly the sign of the frame shift α in Eq. 2) and the readout. Such echoes become even more relevant as one considers stabilizing grid states with larger envelopes, for which the logical qubit is more sensitive to small rotations. With our current stabilization parameters, master-equation simulations reproduce qualitatively the error rate induced by transmon bit-flips as given in the previous paragraph (see Table. 2).

S7 Teleported gates

In this section, we detail a protocol allowing one to perform an arbitrary manipulation of the logical qubit via a teleported gate (18).

In all generality, We consider the case of a target rotation around \mathbf{V} , where $\mathbf{V} = \mathbf{X}, \mathbf{Y}$ or \mathbf{Z} . We can write any state $|\Psi_L\rangle$ of the logical qubit on the $|\pm V_L\rangle$ basis as $|\Psi_L\rangle = \alpha|+V_L\rangle + \beta|-V_L\rangle$.

The teleported gate starts with preparing the transmon in $|+x\rangle$. We then apply a conditional displacement $\mathbf{CD}(\gamma)$, with γ chosen so that $\mathbf{V} = \mathbf{D}(\gamma)$ (same conditional displacement as if measuring \mathbf{V}). Finally, we apply an unconditional displacement $\mathbf{D}(-\gamma/2)$ to recenter the code

(see Sec. S5.4). At the end of the sequence, the joint transmon-oscillator state reads

$$\begin{aligned}
|\psi_{tot}\rangle &= \frac{\alpha}{2}(\mathbf{I} + \mathbf{V})|+V_L\rangle|+x\rangle + \frac{\alpha}{2}(\mathbf{I} - \mathbf{V})|+V_L\rangle|-x\rangle \\
&\quad + \frac{\beta}{2}(\mathbf{I} + \mathbf{V})|-V_L\rangle|+x\rangle + \frac{\beta}{2}(\mathbf{I} - \mathbf{V})|-V_L\rangle|-x\rangle \\
&\approx \alpha|+V_L\rangle|+x\rangle + \beta|-V_L\rangle|-x\rangle,
\end{aligned} \tag{S25}$$

the last equality being strictly verified for an infinitely squeezed state only. We then perform a transmon readout along an axis rotated by (θ, ϕ) from σ_x : the two pointer states of the measurement are $|\theta, \phi\rangle$ and $|\theta + \pi, \phi\rangle$ with the definition $|\theta, \phi\rangle = e^{i\phi/2} \cos \frac{\theta}{2} | + x \rangle + e^{-i\phi/2} \sin \frac{\theta}{2} | - x \rangle$. The Kraus operators acting on the oscillator state depending on the measurement outcome are $\mathbf{M}_{\theta, \phi}$ and $\mathbf{M}_{\theta + \pi, \phi}$ with

$$\mathbf{M}_{\theta, \phi} = e^{i\phi/2} \cos \frac{\theta}{2} | + V_L \rangle \langle + V_L | + e^{-i\phi/2} \sin \frac{\theta}{2} | - V_L \rangle \langle - V_L |. \tag{S26}$$

If we choose $\theta = \frac{\pi}{2}$, these operators are unitary up to a scaling factor. Applying a feedback \mathbf{V} -gate if the transmon is found in $(\theta + \pi, \phi)$, we find that

$$\frac{\mathbf{M}_{\theta, \phi} |\Psi_L\rangle}{\text{Tr}(\mathbf{M}_{\theta, \phi} |\Psi_L\rangle)} = -\mathbf{V} \frac{\mathbf{M}_{\theta + \pi, \phi} |\Psi_L\rangle}{\text{Tr}(\mathbf{M}_{\theta + \pi, \phi} |\Psi_L\rangle)} = \mathbf{R}_V(-\phi) |\Psi_L\rangle \tag{S27}$$

so that the whole sequence performs an unconditional rotation $\mathbf{R}_V(-\phi)$ of the logical qubit around \mathbf{V} by an angle $-\phi$ with

$$\mathbf{R}_V(-\phi) = e^{i\phi/2} | + V_L \rangle \langle + V_L | + e^{-i\phi/2} | - V_L \rangle \langle - V_L |. \tag{S28}$$

Note that in the magic state preparation described in Sec. 6, we choose $\mathbf{V} = \mathbf{X}$ and $(\theta, \phi) = (-\frac{\pi}{4}, 0)$. In all generality, the total evolution is non-unitary. However, applied to the state $| + Z_L \rangle$, when the transmon is measured in $| - \frac{\pi}{4}, 0 \rangle$ (respectively in $| \frac{3\pi}{4}, 0 \rangle$), it rotates it by $3\pi/4$ around \mathbf{Y} (respectively $-\pi/4$ around \mathbf{Y}). In the experiment, we herald the preparation of $|M_L\rangle = \cos \frac{\pi}{8} | + X_L \rangle - \sin \frac{\pi}{8} | - X_L \rangle$ when the transmon is measured in $| - \frac{\pi}{4}, 0 \rangle$, but the preparation could be made deterministic by applying a \mathbf{Y} -gate when the transmon is measured in the orthogonal state.